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Setting the stage for **A dicey proof of the Riemann Hypothesis**
on the basis of **The path of humanity**: *societal innovation for the world of tomorrow* (May 5 2018)
for a good appreciation of this paper

A dicey proof of
the Riemann hypothesis
inspired by societal innovation

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In this paper I present a possible proof of the Riemann Hypothesis. The proof was inspired by a unifying societal philosophy: *Recursive Perspectivism*. Recursive Perspectivism, the proof itself as well as their relations are described in the book "The path of humanity: societal innovation for the world of tomorrow" (in press, 2018; I will refer to it as "the book"), and the presentation "Dicey proofs of the Riemann hypothesis" (December 31, 2017; I will refer to it as "the presentation").
The book is not about number theory. The book *is* about human development, societal innovation and sustainability, and it is founded on a *Recursive Perspectivism* which in turn gives rise to a *recursive multi-actor* interpretation of societal practice. During the writing of "The path of humanity" I slowly came to understand the deep ways in which *The path of humanity*, Bernoulli experiments and the Riemann hypothesis rest on common grounds. Not only do they rest on prime numbers, but furthermore the way in which they develop rests on similar principles. In order to understand these principles better, hesitantly (as I slowly came to understand the imposing reputation of the Riemann hypothesis) I entered the number theoretical realm from the vantage point of Recursive Perspectivism.
The difficulty of understanding whole numbers is in their combined nature: structurally they are multiplications of prime numbers, and numerically they are ordered along the number line. Understanding the interplay between these two viewpoints, structure and content, offers a route to understanding and proving the Riemann hypothesis. I emphasize whole numbers while applying a recursive scheme in my proposal for a proof of the Riemann hypothesis, and I use clean, simple, ancient and well established mathematics in doing so. This would make my approach both elementary and recursive. I use entropic and annihilative arguments from physics. Mathematically I build on Pascal's triangle, Newton's binomial or combinatorial formula, Gauss' normal distribution, Bernoulli experiments and the Mertens function. Be on guard when reading the paper: I am neither a mathematician nor a physicist. I do not claim a high or even a moderate level of proficiency in these fields. I therefore am prone to make errors, and to cut some corners. But even if these warnings would prove to be in due place, the following still would hold true. Pascal, Newton, Gauss, Bernoulli and Mertens offer an imposing foundation for Recursive Perspectivism and the discrete inversely proportional relationship that explains the many *pattern laws* we experience in "our environment". The relations between the Riemann hypothesis, Recursive Perspectivism and societal innovation are important: for our further human development; for a sustainable, a better future. This is the reason why I entered number theory. Therefore I ask you to carefully read this paper, the presentation and the book. Thank you for your attention.

Henk Diepenmaat

This paper is based on The path of humanity: societal innovation for the word of tomorrow, Parthenon Publishers, Almere, The Netherlands (in press, 2018)

Contents

The Riemann hypothesis and the growth of the Mertens function

The Riemann hypothesis is named after Bernhard Riemann, the German mathematician and philosopher (1826-1866) who rather casually mentioned it, and is generally stated in a complex number vocabulary:

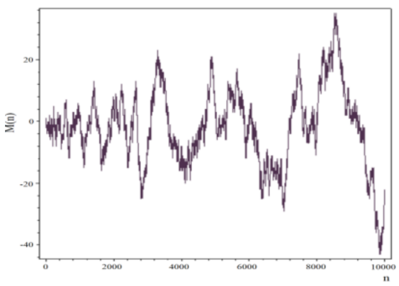
> *"All the nontrivial zeroes of the analytic continuation of the Riemann zeta function ζ have a real part equal to ½."*

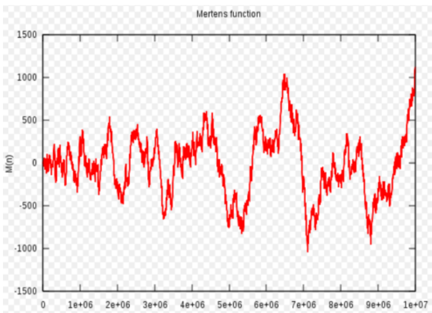
This hypothesis is deeply connected to the *Mertens* function (named after the German mathematician Franz Mertens, 1840-1927), a function built on the *Möbius* function and dealing with positive whole numbers (the realm of Recursive Perspectivism). The Möbius function μ(n) (named after the German mathematician August Ferninand Möbius, 1790-1868) equals 1 for n=1, 0 for squared numbers (a squared number has at least one double prime factor), -1 for square-free numbers with an odd number of prime factors, and +1 for square-free numbers with an even number of prime factors.

The Mertens function M(m) is the sum of the Möbius values (I will call this a "Sigma Möbius", see further on) from 1 to m:

$$M(m) = \sum_{n=1}^{m} \mu(n)$$

The Mertens function is shown below for m from 1 to 10.000 (left, source: Wikipedia) and for m from 1 to 10.000.000 (right, source: Wikimedia). Many people see merely noise. I however see a specifically *interleaved* binomial bell curve (see further).





Edwards (1974, paragraph 12.1), following Littlewood, provides this connection by means of a direct equivalent of the Riemann hypothesis in terms of the Mertens function:

> *"If $M(x) = O(x^{(½ + ε)})$ is true with probability one, the Riemann hypothesis is true with probability one."*

In order to prove the Riemann hypothesis, it would therefore suffice to prove that M(x) grows less rapidly than $x^{(1/2 + ε)}$ for all ε > 0 (see Edwards paragraph 12.1).

Denjoy's probabilistic interpretation of the Riemann hypothesis

As a result of the above, investigating the similarities between the Mertens function and a Bernoulli experiment (f.e. a coin tossing sequence) offers an intriguing possible pathway to proving the Riemann hypothesis. This is explained in Edwards (1974, paragraph 12.3): *Denjoy's probabilistic interpretation of the Riemann hypothesis*, and I will follow this paragraph. In a Bernoulli experiment (for example a coin tossing sequence):

> *"with probability 1 the number of Heads minus the number of Tails grows less rapidly than* $N^{(1/2+\varepsilon)}$*."*

This is because of two reasons: 1) the probability of a Head equals the probability of a Tail and 2) the occurrence of Heads and Tails is independent of each other.

Edwards then argues that it is not altogether unreasonable to assume that in the Mertens function the occurrence of $\mu(n) = +1$ equals the occurrence of $\mu(n) = -1$, and that occurrences of +1 and -1 are independent of each other. If, however, these two assumptions would apply, the conclusion would be that M(x) behaves exactly the same as a Bernoulli experiment. The equivalent statement of the Riemann hypothesis in terms of the Mertens function, at the start of this paper, would then be true. Prove these two not unreasonable assumptions, *and you will have proven the Riemann hypothesis*. This is called Denjoy's probabilistic interpretation of the Riemann hypothesis (after the French mathematician Arnaud Denjoy).

The Denjoy pathway, however, is dicey. Indeed, the Mertens function and a Bernoulli experiment are quite different, the book elaborates on this. The Mertens function on the one hand is completely determined: its graph will be the same over and over. In contrast to this, and on the other hand, each coin tossing sequence will show its own stochastic pathway, asymptotically bound by $N^{(1/2+\varepsilon)}$ but resulting in quite a unique graph. Understanding their similarities is difficult. The Denjoy pathway is dicey indeed.

Stieltjes and Mertens and the Sigma Möbius

Notwithstanding this, the Dutch mathematician Thomas Joannes Stieltjes jr. (1856 - 1894) believed that the most fruitful approach to the Riemann hypothesis was through a study of the growth of M(x) as x -> ∞ (see Edwards, at the end of paragraph 12.1, for a historical account). Stieltjes made a stronger claim than the Riemann hypothesis: $M(x)=O(x^{½})$, which would imply that $M(x)/x^{½}$ remains bounded as x -> ∞. This would prove the Riemann hypothesis. Stieltjes mentioned that he had a proof, and this was generally believed (he was a respected mathematician). He never published such a proof though.

Stieltjes' claim is weaker than *Mertens conjecture*, which states that $|M(x)| < \sqrt{x}$ for all $x > 1$. Mertens' conjecture therefore also would prove the Riemann hypothesis, but is believed to be disproven on the basis of extensive computations with the zeros of the zeta function by Andrew Odlyzko and Herman te Riele in 1985. Stieltjes' claim was considered "highly unlikely" by Odlyzko and te Riele.

Stochastics, interpretation, binomial patterns and annihilation

Probabilistic coin tossing sequences and the completely determined Mertens function may not be equal, but their relations are intriguing nonetheless. In this section I will focus on stochastic (Bernoulli) experiments, for example coin tossing. In the next paragraph I will discuss the discrete inversely proportional relationship. In the paragraph after that I will shift attention to the Mertens function.

Consider the following two statements. We already know:

> *In a coin tossing sequence, with probability 1 the number of Heads minus the number of Tails grows less rapidly* than $N^{(1/2+\varepsilon)}$.

Now I add the following statement:

> *The average of an increasing number of sequences of p coin tosses will approach the binomial pattern of the n-th row of Pascal's triangle better and better.*

The truth of the second statement rests on *entropical* foundations (Boltzmann) and *annihilation* (a physical concept). Take f.e. a sequence of 4 coin tosses (p=4). The table below shows *all 16 possible sequences* (from top to bottom, H=Head, T=Tail, 16 as 2^p=16).

They are grouped together on the basis of the *fractions* of H and T. The percentages of H *linearly* develop as 100% 75% 50% 25% 0% (from left to right), and the percentages of T therefore develop exactly the other way around: 0% 25% 50% 75% 100%.

100	75	50	25	0	% H
H	T H H H	H T H T H T	H T T T	T	
H	H T H H	H T T H T H	T H T T	T	
H	H H T H	T H T H H T	T T H T	T	
H	H H H T	T H H T T H	T T T H	T	
0	25	50	75	100	% T
1	4	6	4	1	(group size, 4-th row Pascal)

Boltzmann and Pascal. This fraction is (these percentages are) a discriminating feature of a sequence of 4 coin tosses as a whole: the sequences may be different from a microscopic view, but they are the same from this macroscopic view (a Boltzmannian argument). This is the argument for the grouping together: for example from the macroscopic point of view of the percentage H or T, the sequences HTTT, THTT, TTHT and TTTH are not different, they are simply the same: they contain 25% H and 75% T. As a result, the *occurrence* of these percentages will follow the *fourth* row of the well-known Pascal's triangle: 1 4 6 4 1 (see the group sizes and the figure below). This is an application of the *Boltzmann principle*.

```
                … 1 …       row 0
              … 1   1 …       row 1
            … 1   2   1 …       row 2
          … 1   3   3   1 …       row 3
        … 1   4   6   4   1 …       row 4
      … 1   5   10  10  5   1 …       row 5
    … 1   6   15  20  15  6   1 …       row 6
  … 1   7   21  35  35  21  7   1 …       row 7
… 1   8   28  56  70  56  28  8   1 …       row 8
```

Annihilation. The number of Heads minus the number of Tails grows less rapidly than $N^{(1/2)+\varepsilon}$. This can be seen as the result of *annihilation* in the p=4 world: an excess of patterns, take THHH (the first pattern of the first group of 4) is annihilated (destroyed, made undone) by other patterns, for example the "inverse" pattern HTTT (the first pattern of the second group of 4), *vice versa*. Any pattern of the first group of 4 would be annihilated by any of the patterns of the second group of 4. Likewise, an excess of H resulting from HHHH would be completely annihilated by one pattern TTTT, or by any four consecutive patterns of the second group of 4. For entropical reasons, repeating a sequence of four coin tosses would approach the fourth row 1 4 6 4 1, and therefore would seek *balance* as a result of this annihilation (*Societal balance* is the central theme of the book. See an internet simulation of a *Galton Board* for an empirical demonstration of balance seeking.)

Note that the six patterns with 50% H and 50% T in the middle would not change an excess number of H and T (or destroy an existing balance). If p is even, the middle group does not matter in this respect, not unlike squared prime factorizations do not matter in the Mertens function. If we would increase the to be repeated sequence of coin tosses to p=5, the fifth row of Pascal's triangle would be the goal the repetition is seemingly aiming for: 1 5 10 10 5 1 (and $2^5=32$):

100	80	60	40	20	0	% H
H	THHHH	TTTTHHHHHH	HHHHTTTTTT	HTTTT	T	
H	HTHHH	THHHTTTHHH	HTTTHHHTTT	THTTT	T	
H	HHTHH	HTHHTHHTTH	THTTHTTHHT	TTHTT	T	
H	HHHTH	HHTHHTHTHT	TTHTTHTHTH	TTTHT	T	
H	HHHHT	HHHTHHTHTT	TTTHTTHTHH	TTTHT	T	
0	20	40	60	80	100	% T
1	5	10	10	5	1	(group size, 5th row of Pascal's triangle)

Merely a matter of interpretation? But repeating a sequence of p coin tosses over and over, say m times, results in a sequence of N=p.m coin tosses. Whether we prefer to look at coin tosses as m times a sequence of p coin tosses, or as only one long sequence of N single coin tosses in a row, or as one massive parallel throw of N dice at the same time (here we touch upon the ergodic hypothesis), merely is a matter of interpretation: the to be executed individual coin tosses will not be influenced, and the actually resulting coin tosses do not change "objectively" when they are being looked at differently. This is because of the two reasons mentioned before: 1) the probability of a Head equals the probability of a Tail and 2) the occurrences of Heads and Tails are independent of each other. These interpretations are exactly the same, providing that N=p.m.

The discrete inversely proportional relationship: two viewpoints

The formula N=p.m implies that p and m are exactly *discretely inversely proportional* with respect to N. Therefore, if we would divide the length of a fixed to be repeated sequence of coin tosses (p) by 2 (or 3 or 4 or …), we would have to multiply the number of repetitions of this sequence of coin tosses (m) by 2 (or 3 or 4 or …), in order to maintain the same bound $N^{(1/2+ \varepsilon)} = (p.m)^{(1/2+ \varepsilon)}$.

Conversely: if we would multiply the length of a fixed to be repeated sequence of coin tosses (p) by 2 (or 3 or 4 or …), we would have to divide the number of repetitions of this sequence of coin tosses (m) by 2 (or 3 or 4 or …), in order to maintain the same bound $N^{(1/2+ \varepsilon)} = (p.m)^{(1/2+ \varepsilon)}$.

For example: 100 repetitions of a sequence of 4 throws would amount to the same whole as 200 repetitions of a sequence of 2 throws, or 400 single throws, or 50 repetitions of a sequence of 8 throws. Here all possible whole number interpretations (solutions) of m.p=400 are presented:

m.p=400

400.1 200.2 100.4 80.5 50.8 25.16 16.25 8.50 5.80 4.100 2.200 1.400

There remains, however, one important difference between N and p.m. One long row of single coin tosses can have any arbitrary length, whereas a repetition of a sequence of p (let's say 4) coin tosses must result in N being a p-fold (a fourfold in the example). As long as N=p.m, this does not matter. If p would be four, we might throw one coin 400 times, or 4 coins 100 times (see the possibilities, the factors, above). In all other cases (so if N=/p.m) a nearby N would _not_ be exactly the same as p.m.

Different possibilities now exist.

If m >> p, the difference between N and p.m would disappear *almost* completely. If p is fixed, and m is getting larger and larger than n, the difference would dwindle away. We may use N as a better and better substitute for p.m, and therefore use $N^{(1/2+\varepsilon)}$ as a better and better substitute for $(p.m)^{(1/2+\varepsilon)}$. (From a philosophical point of view, for a growing m ultimately an illusion of objective reality would emerge, see the book.)

If m is getting equal to p (coming from above), using $N^{(1/2+\varepsilon)}$ as a substitute would become more and more subjective and error-prone. If m << p, this ultimately would result in complete uncertainty.

We may look at this in two different ways: from a statistical point of view, outside-in, by focussing on repetitions, and from a combinatorial point of view, inside-out, by analysing the variability of the repeated pattern.

From a statistical point of view (the first viewpoint, outside-in), in a *stochastic* process like a coin tossing, in order for the boundary $N^{(1/2+\varepsilon)}$ to be valid, the *number of repetitions* m must be sufficiently large with respect to the *variability* in the to be repeated pattern specified by p. We may become more and more confident that this is the case if a certain balance is emerging.

According to combinatorial (entropic, statistical thermodynamic) rules, this *variability within* a repeated combinatorial pattern specified by p is governed by Pascal's triangle (the second viewpoint, inside-out). If the to be repeated pattern with p=n would not show any variability whatsoever, in other words it is completely predetermined and 100 % known, repetition would not be of any help in getting to know this variability any better. If this variability would follow a row of Pascal's triangle, repetition would not show any convergence towards the rows of Pascal's triangle, but would just show this row pattern exactly. Dividing by the number of repetitions would just exactly result in the row itself, after 1, 2, 3 or any number of repetitions.

Consider a dice being thrown only once in front of someone not familiar with dice: this would not result in much insight. And now consider that all the many red dice encountered so far would have six times the number 1 on it. Merely seeing its colour would specify all possible future outcomes of throwing a red dice. Throwing a red dice would result in 1, and therefore the very throw itself would become redundant; obsolete. From a philosophical point of view, this is the way in which a sense of a persistent environment emerges (both physical and mental). I do not need to check whether my front door is still there, when sitting in my living room, and neither does my wife. I might have to check, though, whether my bicycle still is there. (See also the black swan of Karl Popper.)

Mertens function, primorials, the function f_p and binomial patterns

Now let us look at the Mertens function, bearing these two viewpoints in mind. Unlike coin tossing, the Mertens function is not stochastic, but completely determined. Each of the Bernoulli experiment and the Mertens function adheres to one of the the viewpoints. Although the development of the Mertens function therefore cannot equal coin tossing, Heads and Tails are not completely unlike "even" and "odd" square-free numbers (i.e. square-free numbers with an even or an odd number of prime factors).

For the Mertens function we know:

If $M(x) = O(x^{(1/2+\varepsilon)})$ is true with probability one, the Riemann hypothesis is true with probability one.

In the case of a Bernoulli experiment like a coin tossing sequence, we know:

With probability 1 the number of Heads minus the number of Tails grows less rapidly than $N^{(1/2+\varepsilon)}$.

A Bernoulli experiment belongs to the outside-in viewpoint: repeating a process results in insight into the stochastic variability of this process. When throwing a sequence, we will never know this variability for sure (for the complete 100 %), but we may be confident that *the number of Heads minus the number of Tails grows less rapidly than $N^{(1/2+\varepsilon)}$.*

The Mertens function, on the other hand, takes an inside-out viewpoint, as its variability is completely determined. We may be sure that the Mertens function will result in exactly the same graph, over and over, for however large an x we may continue calculating Mertens values.

Primorial numbers and primorial sequences. These two features can be brought into coherence by means of the *primorial sequence*. By consecutively extending the prime factorization with the next prime number in line, starting at 1, we create this primorial sequence (see below). One such an extension I call a *primorial step* (primorial restrictions would go the other way around). Primorial steps result in the next primorial number, and realise the primorial sequence: (1,) 2, 6, 30, 210, 2310, … .

```
2
6=(2 3)                 2.3=6
30=(2 3 5)              2.3.5=30
210=(2 3 5 7)           2.3.5.7=210
2310=(2 3 5 7 11)       2.3.5.7.11=2310

Et cetera.
```

By following the primorial sequence, we are sure not to skip any prime numbers (as *all* the prime numbers so far are being covered). We also are sure not to introduce squared factors (in a primorial number all prime factors are different, a primorial number therefore cannot have a squared factor).

The function $\mathbf{f_p}$. Because of the square-free nature of primorial numbers (duplicate prime factors are completely absent) we can use the combinatorial function for the binomial coefficients (Newton's binomial theorem) to calculate the number of factors of a primorial number consisting of k of the p prime factors. If any set has p *different* elements, the number of different combinations for each k is equal to the *binomial coefficient*:

Comb (p, k) = p! **/** k!(p-k)!

Take for example a set of three different fruits: an apple, a banana and an orange. The possibilities to select different sets of 2 are: (apple banana), (apple, orange), (banana, orange), so three different sets. This number of different sets of two out of these three fruits can be calculated using the combinatorial formula:

Comb (3, 2) = 3! **/** 2!1! = 3

In order to show the way in which this is intimately related to Pascal's Triangle and the binomial bell curve, I use a function $\mathbf{f_p}$. This function calculates the number of all possibilities consisting of k elements, k going from 0 to p, and adds them together:

$\mathbf{f_p}$ =

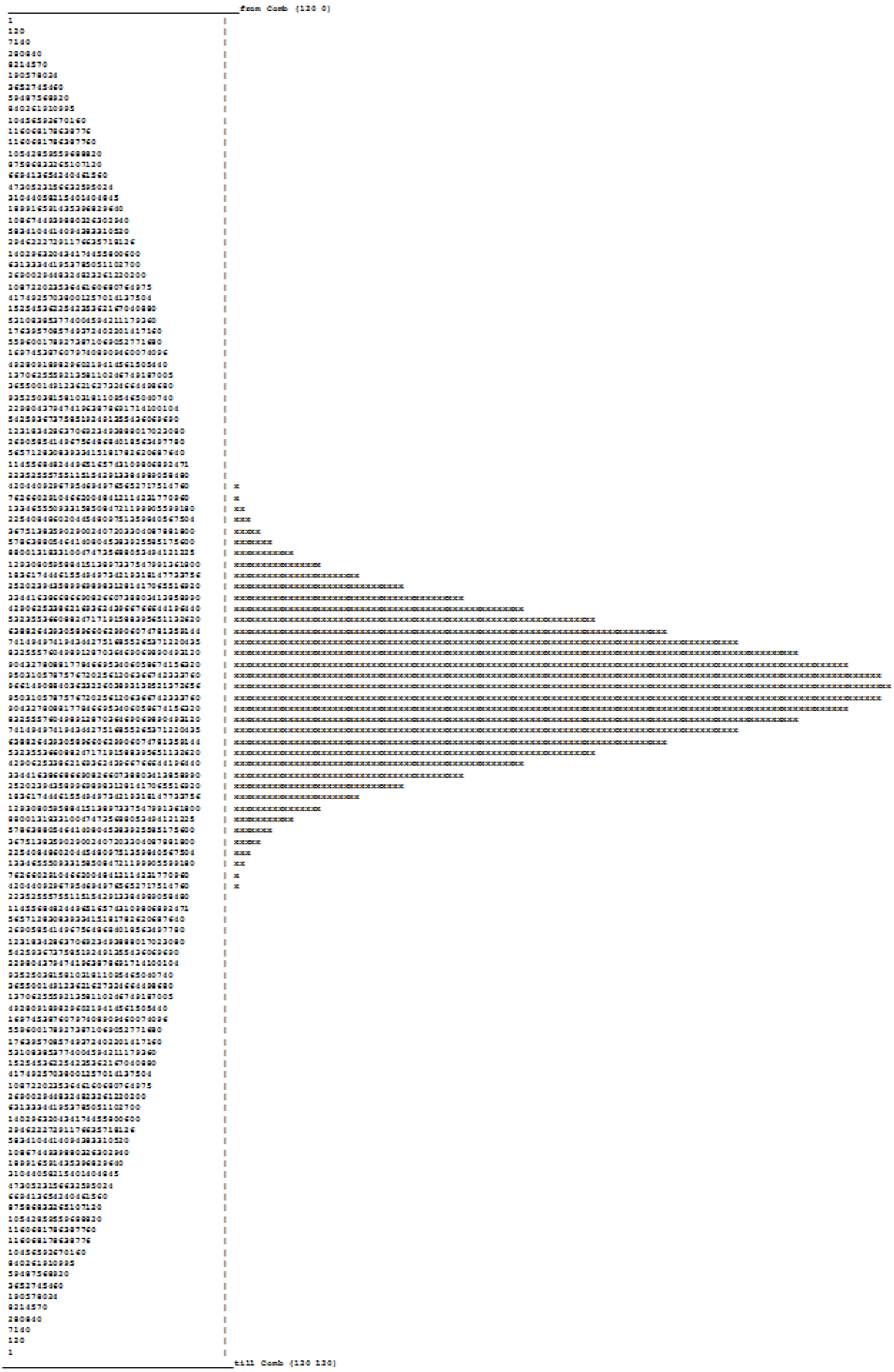
Comb (p, 0) +	(k=0)
Comb (p, 1) +	(k=1)
Comb (p, 2) +	(k=2)
… +	(k=..)
Comb (p, p)	(k=p)

More concisely:

$$\mathbf{f}_{\mathbf{p}} = \Sigma k_{0->p} \text{ Comb } (p, k)$$

Note that Comb (p, 0) results in 1 as the possibility of taking none of the set members exists. Similarly also Comb (p, p) will result in 1, as there is only one way to include all the elements of an unordered set.

This is a convenient function, as writing out all the subsets is rather cumbersome when p is large. Actually this is quite an understatement: the total number of subsets becomes astonishing for large p's. See the p=120 example on the next page: the added result (the sum value) is $2^{\mathbf{120}}$= 1329227995784915872903807060280344576, an enormous amount. It doesn't matter whether it concerns 120 different types of fruit, 120 different types of cars, *or 120 different prime factors*. The p members constituting the set must be different, in the sense of being distinguishable from each other in the Boltzmannian macro sense. Under this condition he resulting pattern will follow the p-th row of Pascal's triangle (the left part of the large picture below is calculated by means of $\mathbf{f}_{\mathbf{p}}$ for p=120, and therefore results in the 120-th row of Pascal's triangle), and the result will be a binomial bell curve.



top value 96614908840363322603893139521372656

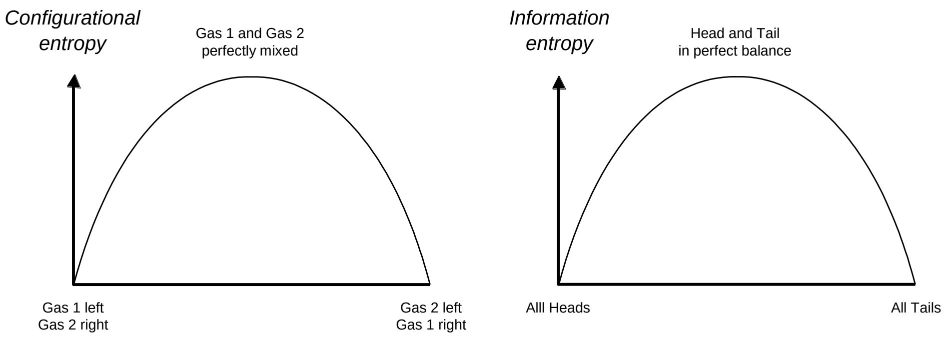
sum value 2^{120}=1329227995784915872903807060280344576

one x is 805124247223947276916044561645568

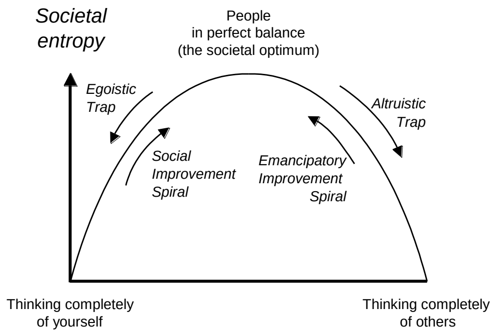
sum/top 13.758

Recursive Perspectivism and Pattern laws. Note that for large p the corresponding row of Pascal's triangle shows a logarithmic pattern (see the left part of the figure above), that is reminiscent of both the configurational entropy when mixing two ideal gasses in chemistry, and Shannon's information entropy in information technology (see figure below). These two entropy curves use the natural log "ln" though; using the natural log instead of a base 10 log on the binomial curve would result in an "entropical" pattern left of the binomial curve in the picture above as well.

In the book, the rather intriguing notions of a *societal entropy* and a *societal balance* are introduced using the same principles, using perspectives as recursive atomic elements. This results in a *societal balance model* (see the third curve in the figure below). The entropic patterns are similar and for a large value for p become more and more the same in quite different activities of human endeavour (natural science, psychology, social science and the largest human activity: society as a whole). These patterns rest on Recursive Perspectivism. Recursive Perspectivism is inherently discrete in nature (perspectives are recursive quants), but high p's will result in a "continuous illusion" for pragmatic reasons. "The path of humanity", the title of the book, is the largest entropical pattern of all. One of the most intriguing aspects of Recursive Perspectivism is that it explains the many mysterious *pattern laws* that we experience as human beings, on the basis of prime numbers: Zipf's Law, Benford's Law, the Pareto principle, the economic law of diminishing returns, the economic distribution of wealth (Piketty), the different economic cycles, natural scientific laws and many more. They all obey the entropic rules and the discrete inversely proportional relationship, and the higher p (the more complex), the better. Recursive Perspectivism offers an Archimedean point that enables overseeing, understanding and explaining these pattern laws, and therefore functions as a unifying philosophy. (See the book).



The societal balance model (see "*The path of humanity*")



Entropy macro state level = ln (number of micro states)

f_p counts factors. When using **f_p** for combining prime factors of a primorial number, multiplication of the resulting sets of prime factors will result in the *factors* of this primorial number. Primorial steps double the number of factors, as this number equals 2^p and each step increases p by 1. The new factors resulting from a primorial step will be added, and are *completely scale invariant* with respect to the already existing factors of the former step, as the new factors simply are the old factors multiplied by the newly added prime factor. This also implies that already existing factors with an odd number of prime factors will be 1-1 accompanied by factors with an even prime factorization, vice versa. As a consequence, the ratio between factors with an odd and an even prime factorization will remain exactly 50%-50%. For example: stepping from 6=(2 3) to 30=(2 3 5) extends the four factors (1 2 3 6) of 6 with the four factors (5 10 15 30), their value is exactly five times (the newly added prime factor) the existing ones, resulting in eight factors: (1 2 3 5 6 10 15 30). The old factors in terms of their number of prime factors were: odd, odd, odd, even; whereas the new factors are: odd, even, even, even, resulting in an equal amount again. The discrete inversely proportional relationship of the new primorial number n=x.y will provide the necessary and sufficient whole number positions for these factors, as before: (1 30)(2 15)(3 10)(5 6)(6 5)(10 3)(15 2)(30 1).

Primorial steps merge the two viewpoints. Primorial steps exhibit both features (both viewpoints) mentioned earlier. They exhibit the notion of ***repetition*** that is present in a

Bernoulli experiment (f.e. coin tossing), as the same procedure is repeated over and over. Primorial steps also exhibit the notion of a ***completely determined variability*** with respect to factors, inherent to combinatorial patterns, as steps are completely determined: a step simply adds (includes) the existing primorial factors, each of them multiplied by the newly added prime number, in a perfectly scale invariant and Möbius inverse way. Primorial steps therefore combine both viewpoints of above, the one dealing with repetition and the one dealing with a determined variability.

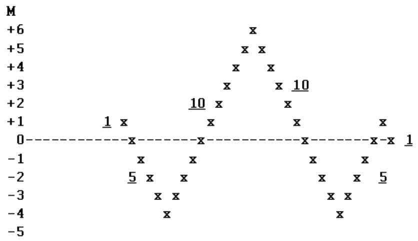
Primorial steps and Pascal's triangle. Primorial steps change factors *both* in terms of distribution over k ranges (binomial structure) *and* in terms of relative order (position) on the whole number line (numerical content). See the three consecutive primorial numbers below. Stepping up is from 210 to 2310 to 30300, stepping down is the other way around. The new factors due to primorial steps down are in italics and underscored. Note that while adding or deleting factors, the steps nicely obey the rows of Pascal's triangle in binomial, structural terms. Also note the symmetries exhibited in these structural patterns of factors (they are rather hidden on the number line due to *interleaving*, the presence of squared numbers and the presence of "strange" square-free numbers, see the presentation and the book, but also see further).

```
210=(2.3.5.7) p=4

   k=0 =>  1      (1)
   k=1 =>  4      (2 3 5 7)
   k=2 =>  6      (6 10 14 15 21 35)
   k=3 =>  4      (30 42 70 105)
   k=4 =>  1      (210)

   2310=(2.3.5.7.11) p=5

   k=0 =>  1      (1)
   k=1 =>  5      (2 3 5 7 11)                                   up: +1   down: -1
   k=2 =>  10     (6 10 14 15 21 22 33 35 55 77)                 up: +4   down: -5
   k=3 =>  10     (30 42 66 70 105 110 154 165 231 385)          up: +6   down: -10
   k=4 =>  5      (210 330 462 770 1155)                         up: +4   down: -10
   k=5 =>  1      (2310)                                         up: +1   down: -5
                                                                          down: -1

   30030=(2.3.5.7.11.13) p=6

   k=0 =>  1      (1)
   k=1 =>  6      (2 3 5 7 11 13)
   k=2 =>  15     (6 10 14 15 21 22 26 33 35 39 55 65 77 91 143)
   k=3 =>  20     (30 42 66 70 78 105 110 130 154 165 182 195 231 273 286 385 429 455 715 1001)
   k=4 =>  15     (210 330 390 462 546 770 858 910 1155 1365 1430 2002 2145 3003 5005)
   k=5 =>  6      (2310 2730 4290 6006 10010 15015)
   k=6 =>  1      (30030)
```

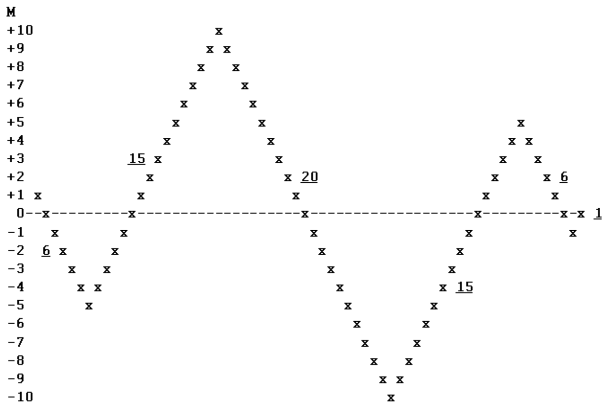
The Sigma Möbius, Interleaving, Sawtooths and structure-content games

Binomial patterns can be represented in a graphical way, using what I call a *Sigma Möbius*. A Sigma Möbius simply is an addition of the values of the Möbius function over a *specified* set of whole numbers. Note that this makes the Mertens function M(m) a specific type of Sigma Möbius: the Sigma Möbius over the range of consecutive numbers from 1 to m.

Sawtooths. I am especially interested in the Sigma Möbius over the *factors* of a primorial number. If I represent the Sigma Möbius over the factors of a primorial number, for example the p=5 primorial number 2310=(2 3 5 7 11), or the p=6 30030=(2 3 5 7 11 13), and I strictly follow the binomial pattern (the corresponding row of Pascal's triangle, the order of the k-ranges), a specific graph results: I call it a *Sawtooth* (see below).



Purely structural sigma Möbius for all square-free numbers with p=5
(1 5 10 10 5 1) 32 square-free factors



Purely structural sigma Möbius for all square-free numbers with p=6
(1 6 15 20 15 6 1) 64 square-free factors

We do *not* need to know the specific value of the prime factors. The number (amount) of them in combination with the requirement of being different suffices. This Sawtooth pattern will be exactly the same for *any* square-free number with the same number of prime factors p (5 and 6 in the example Sawtooth graph), or p different fruits, or whatever. For example, the p=3 primorial Sawtooth of (2 3 5) is exactly the same as the Sawtooth of (2 7 17) or the fruits (apple pear banana).

Note the different types of symmetry for Sawtooths: for p=odd, f.e. 5, a mirror symmetry with respect to the middle line is the result, and for p=even, f.e. p=6, a 180 degrees rotational symmetry with respect to the point in the middle is the result.

Interleaving and the whole number line. In case of a Sawtooth graph of a square-free number, the x-axis is not a regular whole number line. Not only will gaps manifest themselves in between the factors (these factors will be different for different prime factors), but in addition different k-ranges in many cases may (and in case of primorial numbers with p > 3 will) ***interleave***. See for example the factors of the p=4 primorial number, 210=(2 3 5 7), in the order of their binomial pattern (i.e. starting with k=0 and ending with k=3). They do not constitute a well ordered number line:

```
k=0   k=1           k=2               k=3       k=4
(1)(2  3  5  7)(6  10  14  15  21  35)(30  42  70  105)(210)
```

Gaps will manifest themselves. In addition to this, and moreover, when ordering the factors according to the number line, the k=2 6 will take precedence over the k=1 7. Likewise, the k=2 35 will give precedence to the k=3 30. The k-ranges of concern interleave, and as a result the order will not be numerical (content) but binomial (structural). These interleaving processes will be exactly symmetrical, due to the underlying binomial structure and according to the symmetry present in their Sawtooth. They result in a numerical order. Interleaving therefore turns the factors, arranged according to the k-ranges of the binomial pattern (structure):

```
1 2 3 5 7 6 10 14 15 21 35 30 42 70 105 210   (structural order)
0 1 1 1 1 2  2  2  2  2  2  3  3  3   3   4   (k's are ordered)
```

into the range of factors, ordered according to the number line (content):

```
1 2 3 5 6 7 10 14 15 21 30 35 42 70 105 210   (numerical order)
0 1 1 1 2 1  2  2  2  2  3  2  3  3   3   4   (k's are unordered)
```

We can see clearly now that the factors of the primorial number 210 occupy *all* the 16 (as 2^4=16) *whole number blocks on the inversely proportional line* n=x.y, both as x-value and (in reversed order) as y-value, as they are numerically ordered, for all these positions their product equals n, and none are missing:

```
1   2   3  5  6  7  10 14 15 21 30 35 42 70 105 210
210 105 70 42 35 30 21 15 14 10 7  6  5  3  2   1
```

Factors of squared numbers. The formula 2^p for the number of factors holds true for *any* square-free number (or for any set of p distinguishable entities). For a squared number, however, the number of factors (the number of blocks on n=x.y) will be less due to double factors in the binomial expansion according to $\mathbf{f_p}$ (duplicates do not count). See the example of 45=(3 3 5) below, a squared number of p=3. A square-free p=3 number would result in 8 factors, but the squared number 45 has only 6 different factors due to indistinguishable duplicates (a Boltzmannian argument).

45=(3 3 5), p=3 => k=3

k=0 => 1	**(1)**	**1**
k=1 => 3	**(~~3~~ 3 5) => (3 5)**	**2**
k=2 => 3	**(9 15 ~~15~~) => (9 15)**	**2**
k=3 => 1	**(45)**	**1**

Factors: (1 3 5 9 15 45)

The Sawtooth of the squared p=3 number 45 will be symmetrical as before (although it will be shorter than a square-free p=3 number, and with smaller teeth), and the blocks will occupy all the available whole number positions on n=x.y as before. The row 1 2 2 1, however, cannot and does not exist in Pascal’s triangle. Pascal’s triangle is about combinations of sets *without* duplicate members, which in the case of prime factorizations amounts to square-free numbers. Squared numbers can never be equal to square-free numbers, vice versa, as is proven by the *fundamental theorem of arithmetic*, also known as the *unique-prime-factorization theorem*.

Together the squared and the square-free numbers constitute all the numbers on the positive whole number line. This implies that squared numbers would fill in “missing symmetrical rows” of Pascal’s triangle (the triangle is repeated below for convenience). Take for example all p=3 possibilities. They are (the order within the patterns does not matter, and the letters may be substituted with anything at all, including prime factors, the only requirement is that the patterns remains itnact):

AAA	**3:0**	**(all elements are the same, f.e. (3 3 3))**	**1 1 1 1**
AAB	**2:1**	**(one pair and a single one, f.e. (2 3 3))**	**1 2 2 1**
ABC	**1:1:1**	**(all three different, f.e. (2 3 5), row 3)**	**1 3 3 1**

```
            … 1 …       row 0
          … 1  1 …       row 1
        … 1  2  1 …       row 2
      … 1  3  3  1 …       row 3
    … 1  4  6  4  1 …       row 4
   … 1  5  10 10  5  1 …      row 5
  … 1  6  15  20  15  6  1 …      row 6
 … 1  7  21  35  35  21  7  1 …      row 7
… 1  8  28  56  70  56  28  8  1 …      row 8
```

Al the p=5 possibilities, both with duplicates (“squared” in case of p=5 numbers) and without duplicates (“square-free” in case of p=5 numbers, but five different types of fruit would do as well) are:

AAAAA	**5:0**	**(all the same)**	**1 1 1 1 1 1**
ABBBB	**4:1**	**(one and four the same)**	**1 2 2 2 2 1**
AABBB	**2:3**	**(a pair and three the same)**	**1 2 3 3 2 1**
ABCCC	**1:1:3**	**(two different and three the same)**	**1 3 4 4 3 1**
AABCC	**1:2:2**	**(one and two different pairs)**	**1 3 5 5 3 1**
AABCD	**2:1:1:1**	**(a pair and three different ones)**	**1 4 7 7 4 1**
ABCDE	**1:1:1:1:1**	**(all five are different, row 5)**	**1 5 10 10 5 1**

Consider the examples below: 2310, 3125, 6875, 72, 945, 300 and 420, all p=5 numbers but squared differently (2310 is primorial and therefore square-free). For the squared numbers, the actual number of factors will be less than 2^5, as many double factors will be present in the binomial expansion according to $\mathbf{f}_p$. According to the Boltzmann principle, (2 2) and (2 2) cannot be distinguished on the macro level as their product is the same, they are also indistinguishable on the micro level. However, also microscopically different products like (2 5) and (5 2), or (2 2 5) and (5 2 2) and (2 5 2) cannot be distinguished from each other on the macro (the factor) level, as their product is exactly the same. Therefore the number of factors reduces in a predictable, but also complex and somewhat surprising way. See for example the first k=2 factor of 420, this is 4=(2 2). This factor seems to emerge[1] quite unexpectedly, as on the k=1 range of 420 only *one* 2 is to be found. The k=2 factors are calculated on the basis of combining *all the original* prime factors in sets of 2, and not on combining the k=1 factors only. After this combination, the factors are calculated and the double ones are removed. (Mind however that from a recursive perspectivistic point of view one should expect the *probability* of these configurations with "hidden support" to be proportionally higher due to "independent fundaments" ("independent origins", or perhaps is "independent causations" a better term here): they are more likely to emerge.

```
n=2310 primes: (2 3 5 7 11) p=5 => k=5  The p=5 primorial number
   Factors do not need to be corrected: the primorial number is square-free
   k=0 =>  1      (1)
   k=1 =>  5      (2 3 5 7 11)
   k=2 =>  10     (6 10 14 15 21 22 33 35 55 77)
   k=3 =>  10     (30 42 66 70 105 110 154 165 231 385)
   k=4 =>  5      (210 330 462 770 1155)
   k=5 =>  1      (2310)

n=3125 primes: (5 5 5 5 5), p=5 => k=5
   Factors corrected:
   k=0 =>  1      (1)
   k=1 =>  1      (5)
   k=2 =>  1      (25)
   k=3 =>  1      (125)
   k=4 =>  1      (625)
   k=5 =>  1      (3125)

n=6875 primes: (5 5 5 5 11), p=5 => k=5
   Factors corrected:
   k=0 =>  1      (1)
   k=1 =>  2      (5 11)
   k=2 =>  2      (25 55)
   k=3 =>  2      (125 275)
   k=4 =>  2      (625 1375)
   k=5 =>  1      (6875)

n=72 primes: (2 2 2 3 3), p=5 => k=5
   Factors corrected:
   k=0 => 1      (1)
   k=1 => 2      (2 3)
   k=2 => 3      (4 6 9)
   k=3 => 3      (8 12 18)
   k=4 => 2      (24 36)
   k=5 => 1      (72)
```

[1] See "emerging and vanishing properties" in my thesis.

```
n=945 primes: (3 3 3 5 7), p=5 => k=5
   Factors corrected:
   k=0 =>  1     (1)
   k=1 =>  3     (3 5 7)
   k=2 =>  4     (9 15 21 35)
   k=3 =>  4     (27 45 63 105)
   k=4 =>  3     (135 189 315)
   k=5 =>  1     (945)

n=300 primes: (2 2 3 5 5), p=5 => k=5
   Factors corrected:
   k=0 =>  1     (1)
   k=1 =>  3     (2 3 5)
   k=2 =>  5     (4 6 10 15 25)
   k=3 =>  5     (12 20 30 50 75)
   k=4 =>  3     (60 100 150)
   k=5 =>  1     (300)

n=420 primes: (2 2 3 5 7), p=5 => k=5
   Factors corrected:
   k=0 =>  1     (1)
   k=1 =>  4     (2 3 5 7)
   k=2 =>  7     (4 6 10 14 15 21 35)
   k=3 =>  7     (12 20 28 30 42 70 105)
   k=4 =>  4     (60 84 140 210)
   k=5 =>  1     (420)
```

Ordered sawtooths. Now let us redirect our attention to the interleaving of square-free numbers, and primorial numbers as a special case. Consider the following five p=4 square-free example numbers (the fifth is the p=4 primorial number, 210):

```
8756100193      =(293 307 311 313)
46189           =(11 13 17 19)
1938            =(2 3 17 19)
462             =(2 3 7 11)
210             =(2 3 5 7) (the p=4 primorial number)
```

The number of factors must be the same in all cases: 2^4=16, as they are all square-free p=4 numbers. Their Sawtooths, a structural effect, should therefore be exactly the same as well (as of course they are). Their *interleaving* however is different. The level of interleaving is a complex stepwise process, depending on the relative size (the relative order of magnitude) of the prime factors of a number of concern, as all factors are *products* of these prime factors. Interleaving therefore is a typical content related binomial effect, belonging to *numbers* (f.e. *fruits* do not interleave). The relative order of magnitude of the prime factors of the five example numbers is quite different. If we would order their factors according to the number line, and only *after that* draw the Sigma Möbius of these factors, we would take into account the interleaving. The x-axis now is ordered according to the number line. If interleaving is present, the resulting Sawtooth will readily show this as a deviation, an exchange of places with respect to the ideal binomial Sawtooth. The resulting graphs I therefore call *Ordered Sawtooths*, as in an Ordered Sawtooth the factors are interleaved if required. They are ordered according to the number line. This in contrast with the normal or binomial Sawtooths, these simply and blindly follow the binomial pattern of the row of Pascal of concern.

Interleaving cannot shorten a Sawtooth (as squares do): factors are *changing place*, rather than *vanishing*[2]. This may seriously destroy the sharp teeth of the "ideal" binomial Sawtooth pattern according to the rows of Pascal's triangle, as especially the points of the Sawtooth teeth are most prone to interleave: they represent the beginnings and ends of the different k-ranges. The Ordered Sawtooths of the five p=4 square-free numbers are presented below, at right is the p=4 primorial number, 210.

```
8756100193=(293 307 311 313)

fact        |   |
1           |   |X
293         |   X
307         |  X|
311         | XX|
313         |XXX|
89951       | XX|
91123       |  X|
91709       |   X
95477       |   |X
96091       |   |XX
97343       |   |XXX
27974761    |   |XX
28154663    |   |X
28521499    |   X
29884301    |  X|
8756100193  |   X
```

```
46189=(11 13 17 19)

fact        |   |
1           |   |X
11          |   X
13          |  X|
17          | XX|
19          |XXX|
143         | XX|
187         |  X|
209         |   X
221         |   |X
247         |   |XX
323         |   |XXX
2431        |   |XX
2717        |   |X
3553        |   X
4199        |  X|
46189       |   X
```

```
1938=(2 3 17 19)

fact        |   |
1           |   |X
2           |   X
3           |  X|
6           |   X
17          |  X|
19          | XX|
34          |  X|
38          |   X
51          |   |X
57          |   |XX
102         |   |X
114         |   X
323         |   |X
646         |   X
969         |  X|
1938        |   X
```

```
462=(2 3 7 11)

fact        |   |
1           |   |X
2           |   X
3           |  X|
6           |   X
7           |  X|
11          | XX|
14          |  X|
21          |   X
22          |   |X
33          |   |XX
42          |   |X
66          |   X
77          |   |X
154         |   X
231         |  X|
462         |   X
```

```
The p=4 primorial
210=(2 3 5 7)

fact        |   |
1           |   |X
2           |   X
3           |  X|
5           | XX|
6           |  X|
7           | XX|
10          |  X|
14          |   X
15          |   |X
21          |   |XX
30          |   |X
35          |   |XX
42          |   |X
70          |   X
105         |  X|
210         |   X
```

In order to be able to better analyse what is happening, I present the binomial expansion of the factors of the five numbers and their interleaving as well (see next page). For 8756100193 the factors within the different k-ranges seem to flock together, as a direct consequence of their prime factors (293 307 311 313) being highly similar (content). For 46189=(11 13 17 19) this flocking together is still apparent, but less severe, interleaving still is absent. For 1938=(2 3 17 19), the interleaving starts as the prime factors are sufficiently different: the k=2 factor 6 is smaller than the k=1 factors 17 and 19 (k=1

[2] See "emerging and vanishing properties" in my thesis.

numbers are the prime factors). As a consequence of the symmetry of this process, the k=2 factor 323 must be larger than the k=3 factors 102 and 114. (Again, many symmetries show themselves). For 462=(2 3 7 11) the structure of this interleaving is exactly the same as for 1938, although the position of the factors on the number line, their content, is quite different. Apparently, the prime factor ratio boundaries resulting in a different interleaving pattern are not violated. For the primorial number 210=(2 3 5 7), the interleaving changes again. In this case, the flocking together of factors is minimal: the factors are spread *as good as possible* on the line 210=x.y. For a primorial number, all possible whole number positions are in use. Remember: as a primorial number is square-free, also the number of factors is at a maximum for p=4. The primorial number therefore combines the optimal spread with the optimal number of factors, while using a minimal number of perspectives, from the vantage point of Recursive Perspectivism. An imposing structure-content game is at play.

```
n=8756100193 primes: (293 307 311 313), p=4 => k=4

   factors:
   k=0 =>  1     (1)
   k=1 =>  4     (293 307 311 313)
   k=2 =>  6     (                 89951 91123 91709 95477 96091 97343)
   k=3 =>  4     (                             >> 27974761 28154663 28521499 29884301)
   k=4 =>  1     (                                                             >> 8756100193)


n=46189 primes: (11 13 17 19), p=4 => k=4

   factors:
   k=0 =>  1     (1)
   k=1 =>  4     (11 13 17 19)
   k=2 =>  6     (            143 187 209 221 247 323)
   k=3 =>  4     (                                    2431 2717 3553 4199)
   k=4 =>  1     (                                                        46189)

n=1938 primes: (2 3 17 19), p=4 => k=4

   factors:
   k=0 =>  1     (1)
   k=1 =>  4     (2 3   17 19)
   k=2 =>  6     (    6        34 38 51 57          323)
   k=3 =>  4     (                          102 114      646 969)
   k=4 =>  1     (                                                1938)

n=462 primes: (2 3 7 11), p=4 => k=4

   factors:
   k=0 =>  1     (1)
   k=1 =>  4     (2 3   7 11)
   k=2 =>  6     (    6      14 21 22 33         77)
   k=3 =>  4     (                        42 66      154 231)
   k=4 =>  1     (                                          462)

n=210 primes: (2 3 5 7), p=4 => k=4

   factors:
   k=0 =>  1     (1)
   k=1 =>  4     (  2 3 5   7)
   k=2 =>  6     (        6   10 14 15 21    35)
   k=3 =>  4     (                        30     42 70 105)
   k=4 =>  1     (                                         210)
```

From the vantage point of Recursive Perspectivism, we may interpret this wish for filling the inversely proportional line n=x.y as effectively and efficiently as possible, as a high potential *fitness*. A high configurability of perspectives of concern amounts to a potentially high efficaciousness. Primorial numbers of perspectives exploit this configurability to its maximum: blocks are spread optimally, and the number of factors is at its maximum for this number of perspectives. Recursive Perspectivism appreciates a level of configurability as high as possible, for a number of perspectives as limited as possible, as a high aptness for creating value (for realising improvement potential). The book elaborates on this.

Boundaries of k-ranges of primorial numbers. The interleaving of the k-ranges of primorial numbers is limited by well-defined boundaries. These boundaries find their origin in the prime factors of the primorial numbers. In order to appreciate this better, firstly look at the repeated k-factors of the p=6 primorial number 30030=(2 3 5 7 11 13) below, and especially the underlined factors at the *beginning* of the k-ranges: 2 6 30 210 2310 30030. *They constitute the primorial sequence*. They provide the boundaries for higher k-ranges to interleave to the left on the number line. The k=2 range, for example, starting with 6, cannot interleave further to the left than position 6 at the number line, and the k=5 range has an interleaving boundary of 2310.

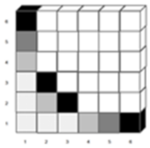
```
factors:
k=0 =>  1     (1)
k=1 =>  6     (2 3 5 7 11 13)
k=2 =>  15    (6 10 14 15 21 22 26 33 35 39 55 65 77 91 143)
k=3 =>  20    (30 42 66 70 78 105 110 130 154 165 182 195 231 273 286 385 429 455 715 1001)
k=4 =>  15    (210 330 390 462 546 770 858 910 1155 1365 1430 2002 2145 3003 5005)
k=5 =>  6     (2310 2730 4290 6006 10010 15015)
k=6 =>  1     (30030)
```

Secondly and likewise, look at the underlined factors at the *end* of the k-ranges. *They constitute the "inverse" of a primorial sequence*. They provide the boundaries for lower k-ranges to interleave to the right on the number line. The k=2 range, for example, ending with 143, cannot interleave further to the right than position 143 at the number line,

```
13
143 = 13.11
1001 = 13.11.7
5005 = 13.11.7.5
15015 = 13.11.7.5.3 and
30030 = 13.11.7.5.3.2
```

The ranges open up at 1 for k=0, and close down at 300030 for k=p, as 2.3.5.7.11.13 = 13.11.7.5.3.2. A primorial sequence follows a recursive process: the boundaries therefore will apply recursively as well. For each primorial step the interleaving effects are bounded as described above, and therefore local in this specific sense. Interleaving is limited in a strict and formal way

The magnificent $x^{1/2}$. Half of *all* the factors of a primorial number will be below $x^{1/2}$, the other half will be above $x^{1/2}$, as a consequence of the discrete inversely proportional relation and the resulting diagonal symmetry in the corresponding black blocks figure. For example, for 6=(2 3) the 2^2=4 blocks are 1.6, 2.3, 3.2 and 6.1, the "whole" positions on the inversely proportional line 6=x.y (see the black blocks figure for 6, below).



Prime numbers (like 5 or 313) have only two "stepping stones" to offer when walking the line n=x.y, on both axes: 5, 1 and 1, 5 and 313, 1 and 1, 313 respectively. Primorial numbers offer the most convenient, the most both effective *and* efficient "stepping stones" when travelling the line n=x.y. Square-free numbers that are not primorial require larger jumps. Squared numbers miss stepping stones (factors) on the basis of the tantalizing scheme presented earlier.

For a primorial number, $x^{1/2}$ cannot be a whole number, as its prime factorization is square-free. Due to the *discrete inversely proportional line* n=x.y it must be true, however, that multiplying the two middle numbers results in n. And indeed, for the 30030 example above, multiplying 165 and 182 results in 30030. Likewise, in the 210 example presented earlier, multiplying 14 and 15 (or 15 and 14) results in 210:

```
1   2   3  5  6  7  10 14 15 21 30 35 42 70 105 210
210 105 70 42 35 30 21 15 14 10 7  6  5  3  2   1
```

When looking back to the factor expansions according to $\mathbf{f_p}$ of the p=4, 5 and 6 primorial numbers 210, 2310 and 30030, two pages back, multiplying the middle two factors, as seen from a strictly binomial Sawtooth point of view, results in:

```
p=4   210=(2 3 5 7):           14.15=210
p=5   2310=(2 3 5 7 11):       77.30=2310
p=6   30030=(2 3 5 7 11 13):   165.182=30030
```

When p=even, multiplication of the two middle numbers results in the primorial number. The two middle numbers are the two factors approaching $x^{1/2}$ best. When p=odd, multiplication also results in the primorial number. But in these cases the first middle number (for the p=5 case, 77) is much larger than the second middle number (30).

The reason is that, in the binomial expansion, the factors are not yet interleaved. I repeat the p=odd expansion of 2310 for convenience (you might want to review the p=even expansions of 210 and 30030 as well, see a few pages up):

```
2310=(2.3.5.7.11) p=5

k=0 =>  1     (1)
k=1 =>  5     (2 3 5 7 11)
k=2 =>  10    (6 10 14 15 21 22 33 35 55 77)
k=3 =>  10    (30 42 66 70 105 110 154 165 231 385)
k=4 =>  5     (210 330 462 770 1155)
k=5 =>  1     (2310)
```

The *inversely proportional relationship* n=x.y orders the factors according to the number line (it is a pattern *after interleaving*). For p=4 and p=6 and every other primorial number with an *even* number of prime factors, the Sigma Möbius will exhibit a rotational symmetry, as shown by their Sawtooths. As a result, the two middle numbers will not change position because of interleaving. For p=5 and every other primorial number with an *odd* number of prime factors, the Sigma Möbius will exhibit a mirror symmetry, as exhibited by their Sawtooth.

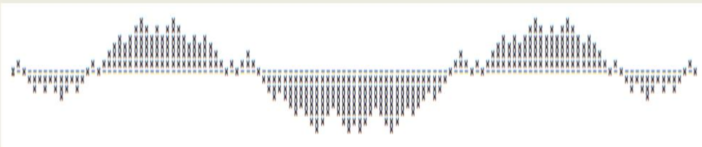
Take the p=5 example: k will increase from 0 to 5. The mean k-value therefore is not a permitted k-value: 5/2=2½ (whole numbers are required). The two middle k-values therefore are the result of rounding down 2½, resulting in k=2 (the "floor"), and rounding up 2½, resulting in k=3 (the "ceiling"). The highest factor on the k=2 range (its upper bound) is 77, and the lowest factor on the k=3 range (its lower bound) is 30, and 77.30=2310.

The teeth of the Sawtooths in general are most prone to interleaving, as they represent the boundaries of the k-ranges. For p=odd, the middle tooth (there is one middle tooth for p=odd) of the Sawtooth is the biggest in terms of Sigma Möbius, and as a consequence this tooth tip will show the most intensive interleaving. As a result of their boundary nature, the two middle numbers of the p=5 primorial number will be *most prone* to interleaving. The reason is that they represent the *largest* factor on the k-range resulting from taking the floor of p/2 (in the example the floor of 5/2, resulting in the k=2 range), and the *smallest* factor on the k-range resulting from taking the ceiling of p/2 (in the example the ceiling of 5/2, resulting in the k=3 range).

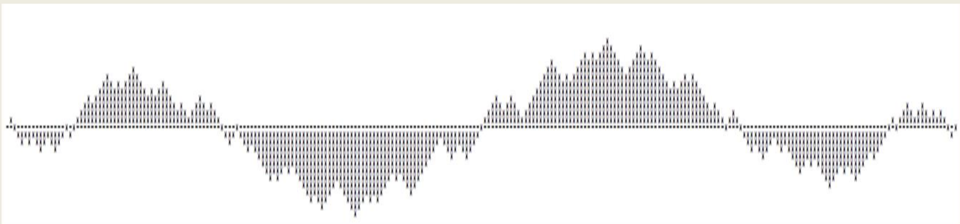
For p=even, the middle position of the Sawtooth is at Sigma Möbius value 0, and in the middle of the k-range possessing most factors. As a consequence, it is most inert in terms of interleaving. As a consequence of this, for p=even, the two middle Sawtooth factors are exactly the same as the two middle Ordered Sawtooth factors. For example for the p=6 primorial number, **165.182 = 165.182 = 30030**. For p=odd, on the other hand, the two middle Sawtooth factors will be quite different from the two middle Ordered Sawtooth factors, and even more so if p is large. For example for the p=5 primorial number, **77.30 = 42.55 = 2310** (before interleaving, 77 and 30 are the two middle factors, and after interleaving, 42 and 55 will be the middle factors, in both cases their product must be 2310 because of the symmetry). For p=13 the difference between the two middle factors *before* and *after* interleaving is even more pronounced: **4199.1155 = 2145.2261 = 4849845.**

Quantum wave patterns

When looking at the Ordered Sawtooth of the p=4 primorial, 210, you might see the emergence of a very typical pattern, well known in quantum physics: a wave pattern. For the p=4 primorial this might not be very convincing yet, but the Ordered Sawtooths of higher primorial numbers like p=7 and p=8 readily reveal their secret (7 and 8 teeth are present respectively, when neglecting the much smaller "in between" teeth):



510510=(2 3 5 7 11 13 17) p=7 2^7 = 128 factors



9699690=(2 3 5 7 11 13 17 19) p=8 2^8 = 256 factors

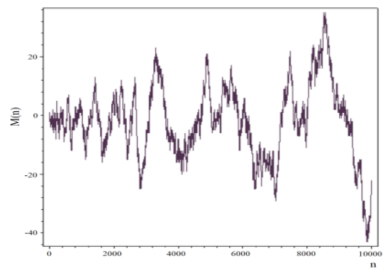
Note that the basic Sawtooth patterns shape these wave patterns (as a consequence the symmetry is different for even and odd numbers of prime factors), and the interleaving turns them into the Ordered Sawtooth wave patterns. The highest peaks (teeth) of the Sawtooth patterns of primorial numbers are most heavily replaced when ordering them according to the number line: they consist of the highest lower k factors and the lowest higher k factors, and therefore they are most likely to interleave. When making a primorial step, teeth interleave within their own boundaries, it may be compared with the eroding of a sand castle at the beach, and this tantalizing process results in the typical quantum waves. Indeed they rest on a quantum process, as factors are only allowed to fill the inversely proportional line n=x.y with *whole* number x and y values (perspectives act like quants). When making primorial steps, f.e. from 6=(2 3) to 30=(2 3 5) to 210=(2 3 5 7) to 2310=(2 3 5 7 11), all the non-primorial numbers in between the numbers of the primorial sequence will obey and fill *their* specific non-primorial formula n=x.y with factors as well.

Back to Riemann and Mertens: three categories of numbers

Now back to the Riemann hypothesis and the Mertens function. Edwards (1974, paragraph 12.1), following Littlewood, provides a direct equivalent to the Riemann hypothesis in terms of the Mertens function, and I repeat:

> *"If $M(x) = O(x^{(1/2+\varepsilon)})$ is true with probability one, the Riemann hypothesis is true with probability one."*

It would therefore suffice to prove that M(x) grows less rapidly than $x^{(1/2+\varepsilon)}$ for all ε > 0, in order to prove the Riemann hypothesis. The characteristic Mertens function is repeated below for m up to 10.000 (source: Wikipedia). Many people see merely noise. I already mentioned that I do not see noise. I see the first part of a specifically interleaved binomial bell curve.



A crucial key to understanding the relevance of primorial steps and interleaving for proving the Riemann hypothesis is the definition (perhaps the distinction) of *three categories* of numbers *from 1 to a (any) primorial number*:

1. *Factors* of this primorial number (square-free, see the sigma Möbius wave pattern)
2. Squared numbers (they are Möbius 0)
3. Square-free non-factors

From here on, I will use "cat" as an abbreviation of "category", f.e. cat 1 is short for category 1.

A number is either not categorized yet (it is bigger than the primorial number of concern), or it is either cat 1 or cat 2 or cat 3. In combination the three categories must cover all the numbers from 1 to the primorial number, even if we do not know where they hide themselves. All three categories in combination play imposing games. Note that cat 2 is fixed (a number is either squared or not, no primorial step can alter this), whereas cat 1 and 3 are interchangeable in specific ways, depending on the primorial number of choice and whether we are primorially stepping up or down. The number 1 is always cat 1.

The three categories allow for using recursive schemes (in line with Recursive Perspectivism). The segment of the number line from 1 to a primorial number I will call the *primorial segment* of this primorial number. *Primorial steps, f.e. from 30=(2 3 5) to 210=(2 3 5 7):*

- Increase the primorial segment by a factor equal to the newly added prime factor (in the example this factor is 7, and 210=30.7);
- Double the number of cat 1 numbers (as the number of factors equals 2^p and p increases with 1);
- Recursively turn some cat 3 numbers of the original primorial number (30 in the example) into cat 1 numbers of the new primorial number (210 in the example), on the basis of the "incorporation" of the new prime number (7) in the prime factorization;
- Introduce new cat 3 numbers for the new primorial number (in this case between 30 and 210).

Due to the recursiveness of the primorial sequence, the original cat 1 numbers (the factors) will remain cat 1 for the new primorial number as well. Cat 2 numbers (the numbers with a squared prime factorization) will always be cat 2 anyhow, for both the original and the new primorial number (they cannot change). So in effect during a primorial step some cat 3 numbers are turned (converted) into cat 1 numbers on the original primorial segment (in the example between 1 and 30), and new cat 3 numbers are created on the new part of the primorial segment (in the example between 30 and 210). The converted numbers were "strange" before on the basis of the newly added prime factor, as introducing this new prime factor turns them from cat 3 into cat 1. The new cat 3 numbers were not yet categorized, and contain at least one prime factor that is not on the prime factorization of the new primorial number.

Stochastics, predetermination, and the binomial bridge

In this paragraph, firstly I will explain the way in which the Mertens function and the number of Heads minus the number of Tails of a coin tossing sequence might be the same, notwithstanding their large differences. After that, I will introduce the *condensation area* and the *free zone*: two special parts of any primorial segment as far as interleaving, the Riemann hypothesis and the Mertens function are concerned. On the basis of this, the crucial steps in proving the Riemann hypothesis can be made.

Where stochastics and predetermination meet. The seemingly stochastic nature of predetermined prime numbers has baffled many mathematicians (see for example the Denjoy interpretation of the Riemann hypothesis). The number of Heads minus the number of Tails of a coin tossing sequence is not unlike the Mertens function: the Sigma Möbius over the square-free numbers. They appear to be strangely similar and extremely different at the same time.

They are strangely similar in that the agreement between the appearance of square-free numbers with an even or an odd prime factorization on the one hand, and the appearance of Heads and Tails in a coin tossing sequence on the other is quite appealing. The Denjoy interpretation of the Riemann hypothesis would require a proof of two statements: the occurrence of $\mu(n) = +1$ equals the occurrence of $\mu(n) = -1$, and occurrences of +1 and -1 are independent of each other. Prove these two statements, and you will have proven the Riemann hypothesis. Primorial sequences clearly show, however, that many of the square-free numbers are *dependent* of each other. And indeed, so far the Denjoy interpretation has not offered a proof of the Riemann hypothesis yet.

At the same time they are extremely different, as a coin tossing sequence is completely and utterly stochastic, whereas the Mertens function is completely and utterly predetermined. A larger difference is difficult to conceive.

In case of a count tossing sequence, we do have to go the whole nine yards in order to precisely know *the number of Heads minus the number of Tails* of a particular sequence of tosses. The reason is that the coin tossing sequence is a stochastic procedure. We know, however that, in a coin tossing sequence, with probability 1 the number of Heads minus the number of Tails grows less rapidly than $N^{(1/2+ \varepsilon)}$.

In case of the Mertens function, it would appear that here also we would have to go the whole nine yards in order to know M(m). But do we really?

Earlier in this paper I have discussed the following sentence:

> *The average of an increasing number of sequences of p coin tosses will approach the binomial pattern of the n-th row of Pascal's triangle better and better.*

This sentence is interesting, as it highlights the role of binomial patterns in coin tosses. Binomial patterns are rows of Pascal's triangle, and each row constitutes a binomial bell curve. For high p, the binomial bell curve is the underlying pattern of the Gauss curve, which rules in stochastics. We also know that the Sigma Möbius of the *cat 1 numbers* (the factors) of a primorial number will follow a strict binomial bell pattern. This binomial bell curve therefore provides a potential bridge, a similarity between Sigma Möbius and coin tosses.

For primorial numbers, the function $\mathbf{f}_p$ presents the appropriate binomial pattern row, and the Sigma Möbius of the *cat 1 numbers* (the factors) of a primorial number will therefore be 0. Actually, for primorial numbers with an even prime factorization, the Sigma Möbius will be 0 already at half the cat 1 numbers (the factors)! See the Sawtooth graphs above. Due to this similarity (this bridge), the following must hold true:

> *The Sigma Möbius over the cat 1 numbers of a growing primorial sequence will grow less rapidly than $x^{(1/2+\varepsilon)}$ for all $\varepsilon > 0$.*

Cat 1 numbers will interleave, but this doesn't alter this fact: interleaving merely re-orders the Sawtooth Möbius values according to the number line, but does not *change* these Möbius values. This interleaving will obey primorial k-range boundaries recursively, and therefore will respect the boundaries of consecutive primorial numbers. Reordering therefore will not alter (or at least will not make unbound) variability on the long run.

In case of the cat 1 numbers (the factors) of the p=n primorial number, we do *not* have to go the whole nine yards in order to know that the Sawtooth pattern will be binomial according to the n-th row of Pascal's triangle, and that the exact outcome of the Sigma Möbius will be 0. The reason is that, as soon as the prime factors determining the primorial number are known, the sequence of factors is completely and 100% predetermined. The pattern therefore must be binomial, as shown by the Sawtooth, and the exact outcome of the Sigma Möbius over the factors (cat 1) must be 0.

The prime numbers and the factors are predetermined and implied. The very moment that we understand the procedure "addition" for whole positive numbers as connecting whole segments on the number line, and "multiplication" as "repeated addition", the prime numbers are completely predetermined constructively and recursively, albeit implicitly. We only have to multiply (to repeatedly add) the prime factors of a primorial number so far, allowing for squares. The first "vacancy" on the number line that cannot be "filled in" in this way must be the <u>next</u> prime factor, required for calculating the following primorial number. A prime number by definition cannot be constructed by means of multiplying smaller whole numbers, and this procedure therefore offers the constructive definition of prime numbers. After finding a new prime number, the procedure can be repeated recursively, including the new (the lastly added) prime number.

As soon as the primorial prime factors of a new primorial number are known, the complete category 1 number set (the factors) up to and including the new primorial number is fixed and determined. Their binomial Sawtooth structure is implicitly known, as is the resulting Sigma Möbius.

Proof spoilers, the condensation area and the free zone

We have established two things now:

1: The Sigma Möbius over the cat 1 numbers (the factors) of a primorial number is 0. For primorial numbers with an even prime factorization, the Sigma Möbius over half the cat 1 numbers is 0.

2:. The Sigma Möbius over *the cat 1 numbers of a growing primorial sequence* will grow less rapidly than $x^{(1/2+ \varepsilon)}$ for all ε > 0.

However, this does not suffice for our purpose: proving the Riemann hypothesis. In order to do this, the Sigma Möbius over _all_ the numbers (which is the Mertens function) should grow less rapidly than $x^{(1/2+ \varepsilon)}$ for all ε > 0.

The crucial step: getting rid of proof spoilers. Remember that cat 2 numbers cannot be turned into cat 1 or cat 3 numbers: they simply are what they are. This identifies cat 3 numbers as the *proof spoilers* of the Riemann hypothesis: they prevent the establishment that M(x) grows less rapidly than $x^{(1/2+ \varepsilon)}$ for all ε > 0. They also prevent the Mertens function from 1 to a primorial number (the Sigma Möbius from 1 to a primorial number) from becoming 0 (which essentially is the same).

See for example the primorial segment from 1 to 30=(2 3 5), below. On the second row, the three categories of the numbers are specified. On the third row, the value of the Mertens function is provided.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	1	1	2	1	1	3	2	2	1	3	2	3	3	1	2	3	2	3	2	3	3	3	2	2	3	2	2	3	1
1	0	-1	-1	-2	-1	-2	-2	-2	-1	-2	-2	-3	-2	-1	-1	-2	-2	-3	-3	-2	-1	-2	-2	-2	-1	-1	-1	-2	-3

The value -3 of M(30) at the right end of the third row (the Mertens row) is *completely due* to the category 3 numbers (the proof spoilers), as the sigma Möbius over cat 1 numbers (the factors) of a primorial number equals 0. (The Möbius value of cat 2 numbers, the squared numbers, is 0.)

The crucial step in proving the Riemann hypothesis therefore is dealing with the proof spoilers (the cat 3 numbers). And one way of doing this is getting rid of them.

A careful analysis of the development of categories on primorial segments will shed light on the possibilities of getting rid of the proof spoilers. In this analysis I will emphasise both the start and the end of primorial segments. While developing the primorial sequence, at the start of the consecutive primorial segments a divergently growing "*condensation area*" will develop. Likewise, and for reasons of symmetry, at the end of the primorial segment a divergently growing "*free zone*" will manifest itself. They will be explained and discussed below.

The condensation area. A specific starting range of a primorial segment (the segment from 1 to the primorial number of concern) of any primorial number cannot contain *any* cat 3 number. This cat 3 free starting range I call the *condensation segment* or *condensation area*. It consists completely of category 1 and category 2 numbers.

The German mathematician David Hilbert (1862-1943) introduced the term condensation for the flocking together of prime numbers on different parts of the number line. Perhaps this metaphor was inspired by the way in which for example H_2O vapour molecules condensate into water droplets. It is intriguing that the name *condensation area* is so well in place here, as around 1920-1930 the formalist Hilbert was an opponent of Luitzen Brouwer, a developer and proponent of intuitionism in mathematics. As the outcome of a fierce scientific battle between formalism and intuitionism in mathematics, intuitionism did barely survive. Notwithstanding this, recursive primorial steps and recursive perspectivism seem to fit the bill of intuitionistic mathematics better than formal mathematics, as far as I am able to distinguish these two matters (but remember: I am a reflective pragmatist, not a mathematician).

Cat 1 numbers of a primorial number are flocking together at the very beginning of the primorial segment. The condensation metaphor is even more apt on this segment, as it has a significant philosophical relevance: I use the term directly inspired by its physical meaning in Bose-Einstein condensation. In a similar way that Bose-Einstein condensation causes *superconductivity* of electrons, primorial category 1 condensation causes *superconfigurability* of perspectives. I suspect that (and Recursive Perspectivism suggests that) these two phenomena are deeply akin.

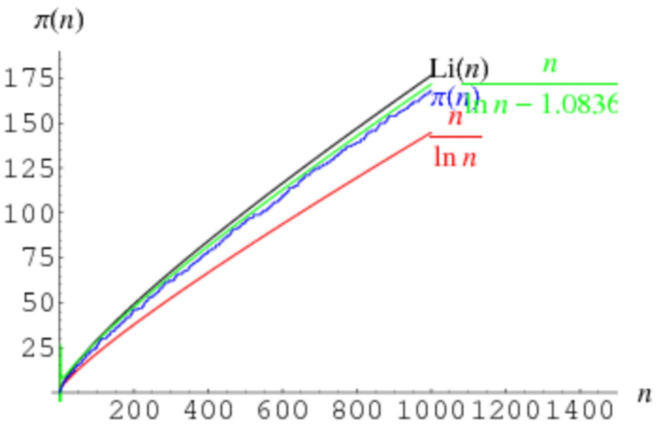
In the condensation area, the category 1 condensation is pushed to its maximum, as category 3 numbers are completely absent (category 2 numbers will always be category 2 numbers: they are "inert" to category changes).

The condensation area of a primorial number of p prime factors *extends*[3] (*overshoots*) the segment from 1 to the largest prime factor of this primorial number: it ranges from 1 to the *anticipated* prime number of the *forthcoming* primorial step *minus 1*. In other words: the (p+1)th prime number minus 1 (remember: the (p+1)th prime number itself is

[3] From a philosophical point of view it is existential (this reminds me of Heidegger's Sein und Zeit).

category 3). The reason is that, as long as no new prime factors are introduced, no new non-factor square-free numbers (i.e. category 3 numbers) are possible. Or the other way around: every square-free number to be encountered up to the *next* prime number will consist of prime factors of the primorial number of concern, and therefore will have to be category 1.

On average and on the long run, the overshoot of the (p+1)th prime number minus 1 is governed by the well-known analytic estimate of the Prime Number Theorem (PNT): the number of primes to n is $\Pi(n) \sim n/\ln(n)$, see the figure below[4].



For example: the category 3 free condensation segment of the primorial numbers:

6=(2 3)	is	5-1=4
30=(2 3 5)	is	7-1=6
210=(2 3 5 7)	is	11-1=10
2310=(2 3 5 7 11)	is	13-1=12
30010=(2 3 5 7 11 13)	is	17-1=16
510510=(2 3 5 7 11 13 17)	is	19-1=18

During a primorial sequence, the condensation segment boundary of the (p+1)th prime number minus 1 must be valid for every primorial number, even if we do not have a clue of the whereabouts of this anticipated (p+1)th prime number and the (p+1)th primorial number on the number line. And there *always* will be such an anticipated next prime number (Euclid proved this). The condensation segment overshoots the last prime factor by a minimum of 1 (this is in case of a twin prime: the lastly added primorial prime is the first of the twin primes, the anticipated prime is the second of the twin primes). Its maximum obviously depends on the prime gap between the p-th and the (p+1)th prime

[4] Source: Mathworld. $\Pi(n) \sim Li(n)$, an even better analytical estimate, firstly discovered by Gauss, see www.mathworld.wolfrahm.com under Prime Number Theorem.

number. Again, the PNT governs asymptotically: for large n, a number is a prime with possibility ~1/ln(n), and the average spacing between the primes to n will be ~ln(n). (These statements are equivalent to the PNT.)

The free zone. The factors x (or y) over the primorial segment of a primorial number n will *obey and fill* the inversely proportial line n=x.y in a quantized way (only whole numbers are allowed). Ordered sawtooths are symmetrical (whole number positions on inversely proportional lines are diagonally symmetrical). A primorial step will therefore simultaneously change the *beginning* (the *condensation area*) *and* the *end* of the new primorial segment with respect to the old one.

The right end (the upper end) of primorial segments therefore is interesting as well. Indeed, there exists a completely category 1 *free zone* between any primorial number and the exact half of this number. The primorial number and its (whole number) half always are the two largest factors of a primorial number, as 2 is a prime factor: for this reason they are category 1, and all the numbers in-between therefore must either be category 2 or category 3.

For example: the category 1 free "free zone" of the primorial numbers:

6=(2 3)	is	3-6 (excluding 3 and 6)
30=(2 3 5)	is	15-30 (excluding)
210=(2 3 5 7)	is	105-210 (excluding)
2310=(2 3 5 7 11)	is	1155-2310 (ecluding)
30010=(2 3 5 7 11 13)	is	15005-30010 (excluding)
510510=(2 3 5 7 11 13 17)	is	255255-510510 (excluding)

During a primorial step, the *condensation area* will be extended with the *forthcoming* prime gap. Likewise, the free zone will be "increased" to half the *new* primorial number.

Bonse's inequality: the natural log and the square root play leapfrog. Bonse's inequality[5] states that if p_1, p_2, …, p_n, p_{n+1} are the first (the smallest) n+1 prime numbers, then $p_1 . p_2 . .. . p_n > (p_{n+1})^2$ for all n > 3. In words: the primorial number must be *larger* than the square of the forthcoming prime number. Similar inequalities exist for higher

[5] A complete elementary proof should be available in H. Bonse (1907), "Über eine bekannte Eigenschaft der Zahl 30 und ihre Verallgemeinerung", Archiv der Mathematik und Physik 3 (12): 292–295.

powers, as can be proven on the basis of Bertrand's postulate[6] (there is always a prime between n and 2n, this was proven in 1852).

As a consequence, a primorial number (the left part of the Bonse inequality) is larger than the square of *the condensation area plus 1* (the right part of the Bonse inequality: remember, the condensation area equals the *forthcoming* prime number *minus* 1). At the same time, subsequent primes to be encountered while making primorial steps will obey the natural logarithm of the Prime Number Theorem (an analytical estimate).

A remarkable feature of the natural log, ln x, is that it increases slower than *any* power, x^a. And the same holds true for *any* power of ln x: the graph of $(\ln x)^N$ will eventually drop below, and for ever after stay below, the graph of x^a, no matter how big N or how small a. John Derbyshire elaborates on this in his popularized book[7] "Prime Obsession", chapter 5, Riemann's zeta function, paragraph IV. Put succinctly: a graph of ln x (or $(\ln x)^N$) will eventually cut *any* x^a curve, and remain below it ever after. This obviously is true for large whole number a's, like 2 or 5: the x^2 curve is known to go sky high quite fast. But, perhaps more surprisingly, it is true for fractional a's, like $x^{½}$, as well, although for really small a's it may take quite a while. As Derbyshire puts it: "You need to go out east to the neighbourhood of $x=7.9414 \times 10^{3959}$ before $(\ln x)^{100}$ drops below the $x^{0.1}$ curve; but eventually it does."

In making primorial steps, an intriguing wheel has been greased. The primorial number of concern will grow with a factor equal to the newly added prime number, on the long run governed by the natural logarithm of the Prime Number Theorem. This implies a logarithmic growth. At the same time the square of the condensation area of this primorial number of concern will approach the *forthcoming* primoral number better and better. Or: the condensation area will approach the square root of the *forthcoming* prime number of the next primorial step better and better. This implies a power growth. While making primorial steps, the natural logarithm and the square root play leapfrog in a fascinating way, keeping each other in balance better and better on the long run, and none of them giving in. This offers an intriguing asymptotic view on the truth of the Riemann hypothesis.

The proof. From the point of view of the Riemann hypothesis, it is very interesting to know the way in which the fractions of cat 1 and cat 3 numbers will develop on the whole

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[6] Mfb, staff mentor of physics forums, replies at a question concerning Bonse's inequality that as a consequence of Bertrand's theorem the following must hold: $p_n p_{n-1} p_{n-2} \cdots > \frac{1}{2} p_{n+1} \frac{1}{2^2} p_{n+1} \frac{1}{2^3} p_{n+1} \cdots$. I quote: "As long as the product of the remaining primes is larger than 64, the product is larger than p_{n+1}^3. That happens for 2*3*5*7, so 2*3*5*7*11*13*17 = 510510 > 6859 = 193 is the first number where the general proof works, but 2*3*5*7*11 > 13^3 is where the inequality starts being valid. It should be obvious how to extend that to larger powers." Reference https://www.physicsforums.com/threads/bonses-inequality.903074/

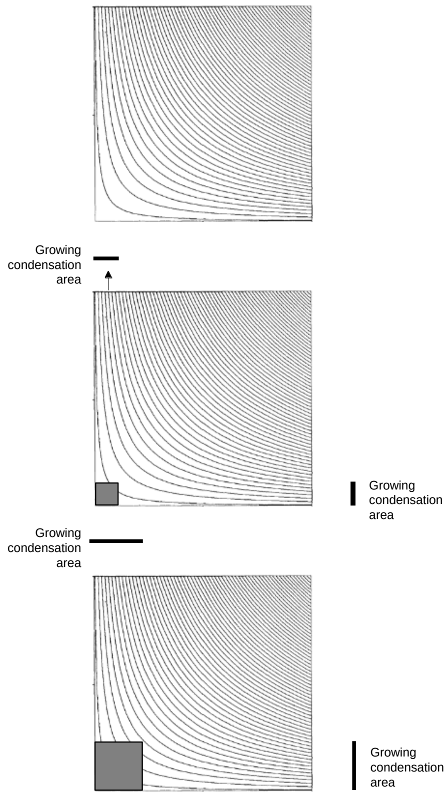
[7] Thank god for high standard popular books, they provide bridges to the disciplinary fortresses of this world!

number line during continual primorial steps. If during the development of the primorial sequence cat 1 would systematically and progressively drive out cat 3, the proof spoilers would disappear, *M(x) = O(*$x^{(1/2+\varepsilon)}$*)* would become true with probability one, and therefore the conclusion would have to be that the Riemann hypothesis is true with probability 1.

And this is exactly what happens. While making primorial steps, the condensation area, an increasingly and divergently growing segment from 1 upward, will not contain *any category 3 numbers whatsoever*. Simultaneously, and for symmetrical reasons, the free zone (free of cat 1 numbers) will develop between the *consecutive* primorial numbers and their halves. A stronger and stronger microscope will be required to be able to notice the condensation area, as it becomes very (very! very!!) small with respect to the growing primorial segment. Nonetheless, the condensation area is a progressively upward moving frontier, and divergent.

As a result, *M(x) = O(*$x^{(1/2+\varepsilon)}$) is true with probability one, and therefore the Riemann hypothesis should be true with probability 1.

The growing condensation area is presented in the figure at the right hand. Due to the symmetry of the discrete inversely proportional line n=x.y, the absence of cat 3 numbers in the condensation area will find its counterpart in the absence of cat 1 numbers in the free zone.



As a result of consecutive recursive primorial steps, from the condensation area to the free zone the number of cat 1 numbers will thin out, converging to (but never quite reaching) 0 % of the square-free numbers, and the number of cat 3 numbers will increase, converging to (but never quite reaching) 100 % of the square-free numbers. Cat 2 numbers will converge to zeta (2), in which zeta is Euler’s zeta function.

While making primorial steps, for any consecutive primorial number its condensation area (a growing and diverging segment at start of the primorial segment) will be completely cat 3 free. As a consequence, the Mertens function on the one hand and the Sigma Möbius over the factors (the cat 1 numbers) of the primorial number of concern on the other hand will be exactly (!) equal for this condensation area. Proof spoilers for the Riemann hypothesis will be absent, and as a result M(x) grows less rapidly than $x^{(1/2+\varepsilon)}$ for all $\varepsilon > 0$. *The Riemann hypothesis therefore must be true*.

We may want to look at this in the following manner. When looking at a graph of the Mertens function M(m) from 1 to m, no matter how large m will be, we will always be looking at (a first part of) the condensation part of a possibly enormous (enormous! enormous!!) primorial number. This primorial number can be calculated by simply multiplying all the prime numbers in this condensation segment graph from 1 to m.

In summary: due to the recursive nature of primorial steps, resulting in a primorial sequence, and the absence of category 3 proof spoilers in the growing and diverging condensation area of the primorial numbers in this primorial sequence, *the Riemann hypothesis must be true*.

Some graphs that illustrate the formal point

Firstly the development of squared and square-free numbers is presented (remember: the fraction of squared numbers converges to zeta (2), but nonetheless this convergence is non-monotonous and rather messy).

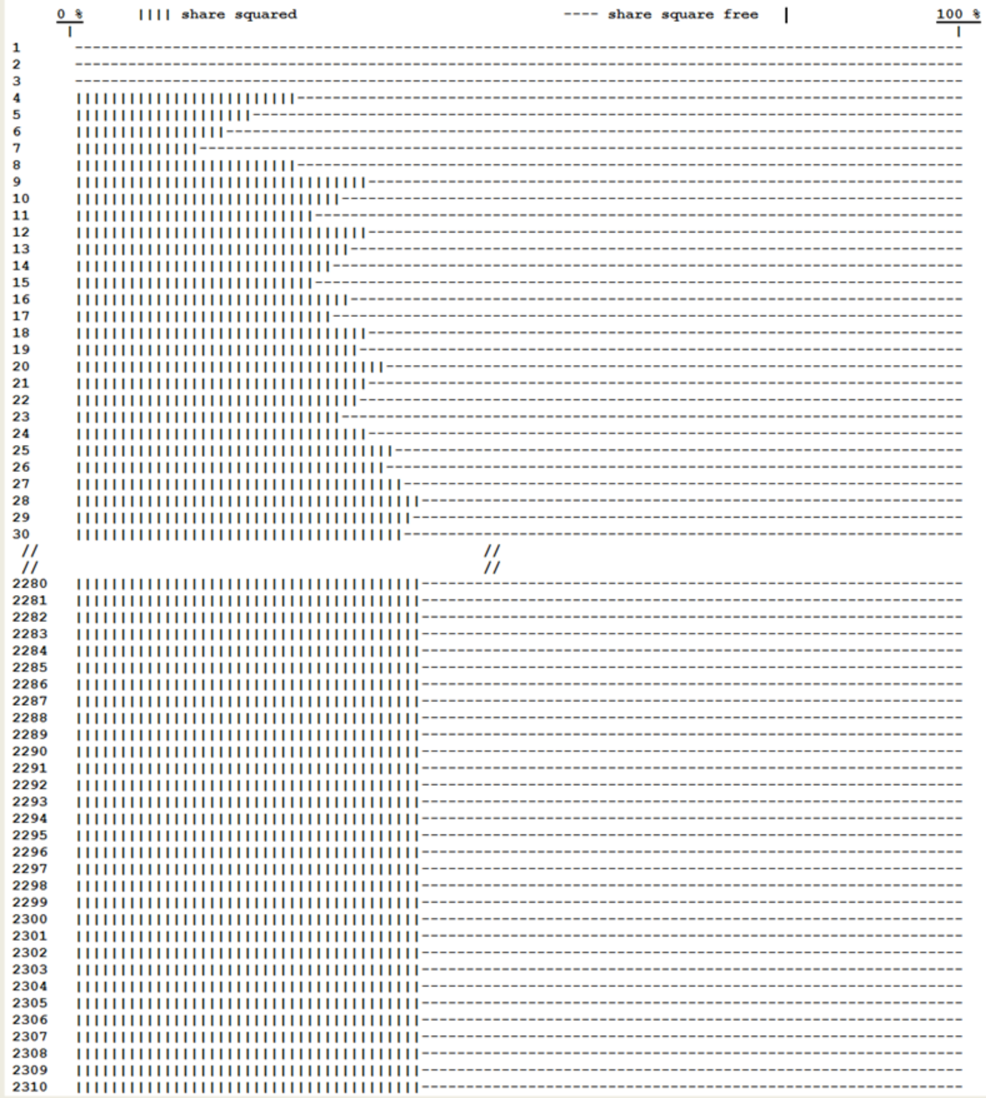
After that, *the growing condensation area* is shown while making primorial steps. Simultaneously these graphs show the thinning out of cat 1 numbers and the thickening of cat 3 numbers as a fraction of the square-free numbers. The condensation area is cat 1 free. The free zone is almost cat 1 free (with the exception of the primorial number m and its half). Category 1 numbers will be *completely* absent in the segment between m and ½m. Due to the symmetry in m=x.y, all the new cat 1 numbers of each step (their total number doubles) minus 2 therefore will fall in between the condensation area of the last primorial step and half the new primorial number. This results in a preference for positions near the diagonal (a direct, albeit discrete, relationship with the central limit theorem and the Lindeberg condition exists, I suspect).

As a result, from the end of the condensation area to the beginning of the free zone, at start cat 1 numbers will dominate and at the end cat 3 numbers will dominate. Due to interleaving this will look like a messy and rather fluctuating process, but recursively the boundaries of the k-ranges of the primorial steps described before will apply. During primorial steps, cat 1 numbers will drive out cat 3 numbers along a moving, albeit diffuse, frontier.

A: Graphs of squared and square-free numbers

Two categories of numbers.

N	share squared rational	share squared decimal	share square-free rational	share square-free decimal
1	0	0.0	1	100.0
2	0	0.0	1	100.0
3	0	0.0	1	100.0
4	1/4	25.0	3/4	75.0
5	1/5	20.0	4/5	80.0
6	1/6	16.66666	5/6	83.33333
7	1/7	14.28571	6/7	85.71429
8	1/4	25.0	3/4	75.0
9	1/3	33.33333	2/3	66.66666
10	3/10	30.0	7/10	70.0
//				
25	9/25	36.0	16/25	64.0
26	9/26	34.61538	17/26	65.38461
27	10/27	37.03703	17/27	62.96296
28	11/28	39.28571	17/28	60.71428
29	11/29	37.93103	18/29	62.06896
30	11/30	36.66666	19/30	63.33333
//				
2280	149/380	39.21052	231/380	60.78947
2281	894/2281	39.19333	1387/2281	60.80666
2282	447/1141	39.17616	694/1141	60.82383
2283	298/761	39.159	463/761	60.841
2284	895/2284	39.18564	1389/2284	60.81436
2285	179/457	39.16849	278/457	60.83151
2286	448/1143	39.1951	695/1143	60.8049
2287	896/2287	39.17796	1391/2287	60.82203
2288	69/176	39.20454	107/176	60.79545
2289	299/763	39.18742	464/763	60.81258
2290	897/2290	39.17030	1393/2290	60.82969
2291	897/2291	39.15321	1394/2291	60.84679
2292	449/1146	39.17975	697/1146	60.82024
2293	898/2293	39.16267	1395/2293	60.83733
2294	449/1147	39.14559	698/1147	60.85440
2295	899/2295	39.17211	1396/2295	60.82789
2296	225/574	39.19860	349/574	60.80139
2297	900/2297	39.18154	1397/2297	60.81846
2298	150/383	39.16449	233/383	60.83551
2299	901/2299	39.19095	1398/2299	60.80904
2300	451/1150	39.21739	699/1150	60.78260
2301	902/2301	39.20034	1399/2301	60.79965
2302	451/1151	39.18332	700/1151	60.81668
2303	129/329	39.20973	200/329	60.79027
2304	113/288	39.23611	175/288	60.76389
2305	904/2305	39.21909	1401/2305	60.78091
2306	452/1153	39.20208	701/1153	60.79792
2307	904/2307	39.18509	1403/2307	60.81491
2308	905/2308	39.21143	1403/2308	60.78856
2309	905/2309	39.19445	1404/2309	60.80554
2310	181/462	39.17749	281/462	60.82251


0 %
|||| share squared
---- share square free
100 %
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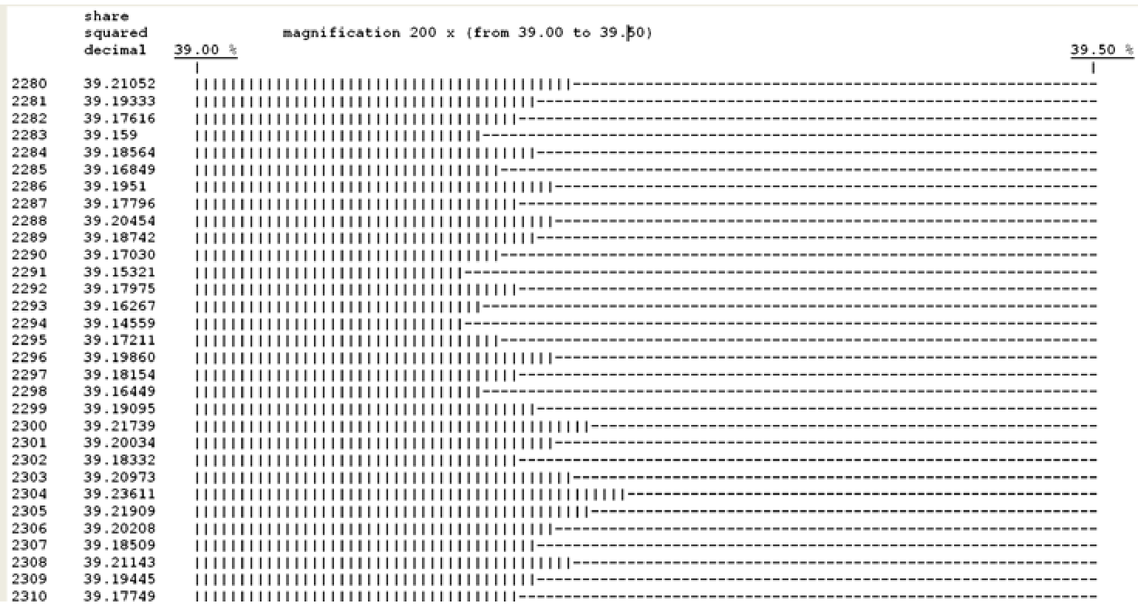
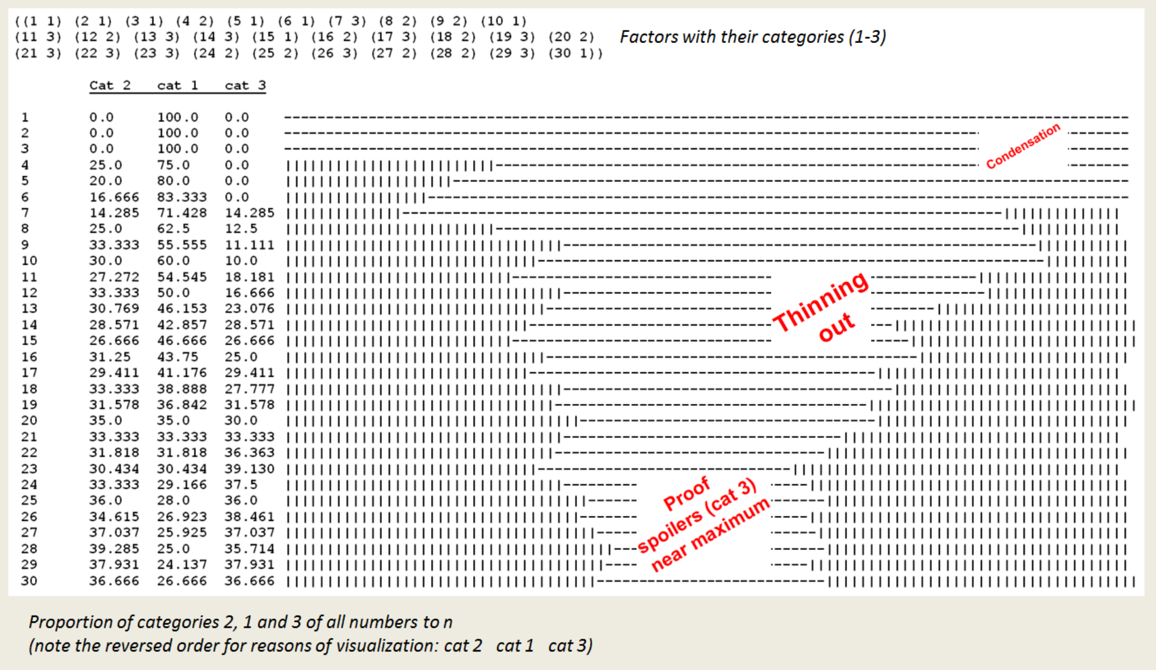
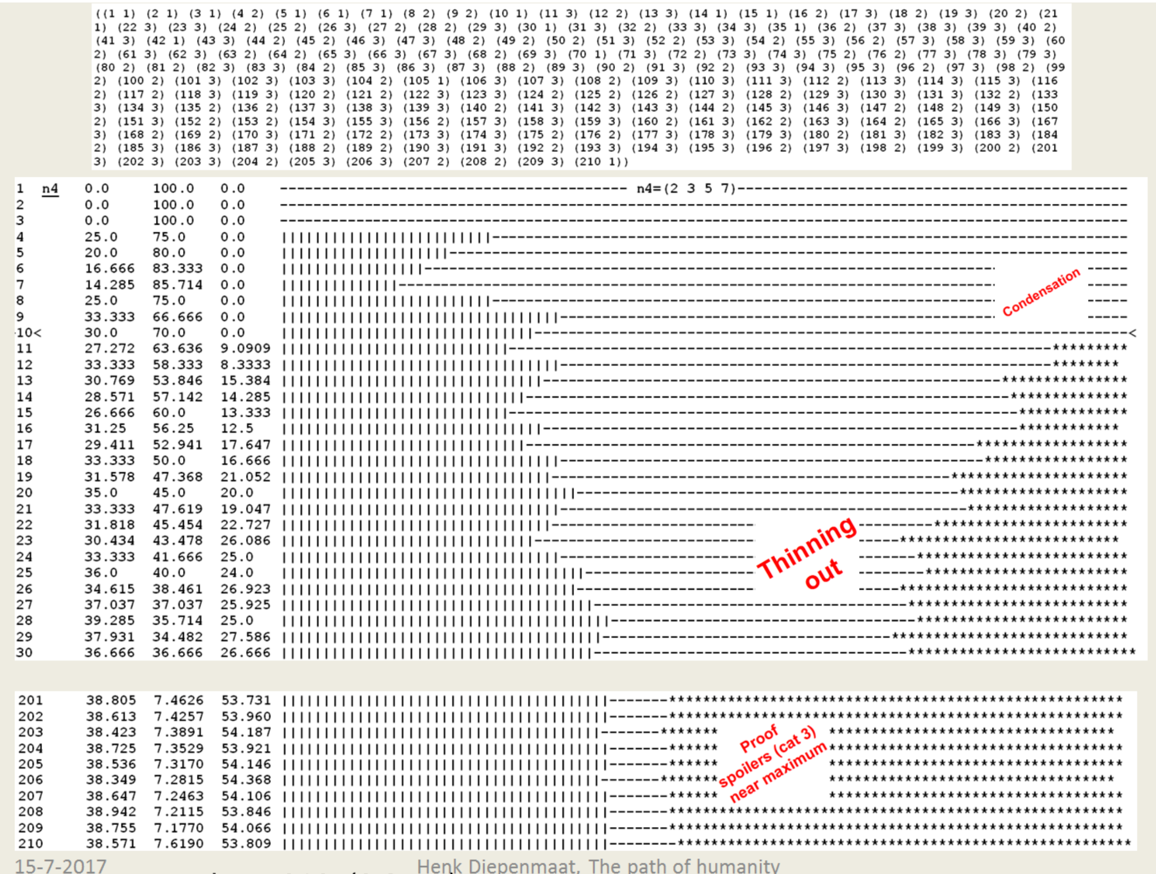

share squared decimal
magnification 200 x (from 39.00 to 39.50)
39.00 %
39.50 %
2280 39.21052
2281 39.19333
2282 39.17616
2283 39.159
2284 39.18564
2285 39.16849
2286 39.1951
2287 39.17796
2288 39.20454
2289 39.18742
2290 39.17030
2291 39.15321
2292 39.17975
2293 39.16267
2294 39.14559
2295 39.17211
2296 39.19860
2297 39.18154
2298 39.16449
2299 39.19095
2300 39.21739
2301 39.20034
2302 39.18332
2303 39.20973
2304 39.23611
2305 39.21909
2306 39.20208
2307 39.18509
2308 39.21143
2309 39.19445
2310 39.17749

B: Graphs of condensation and thinning out: some primorial steps

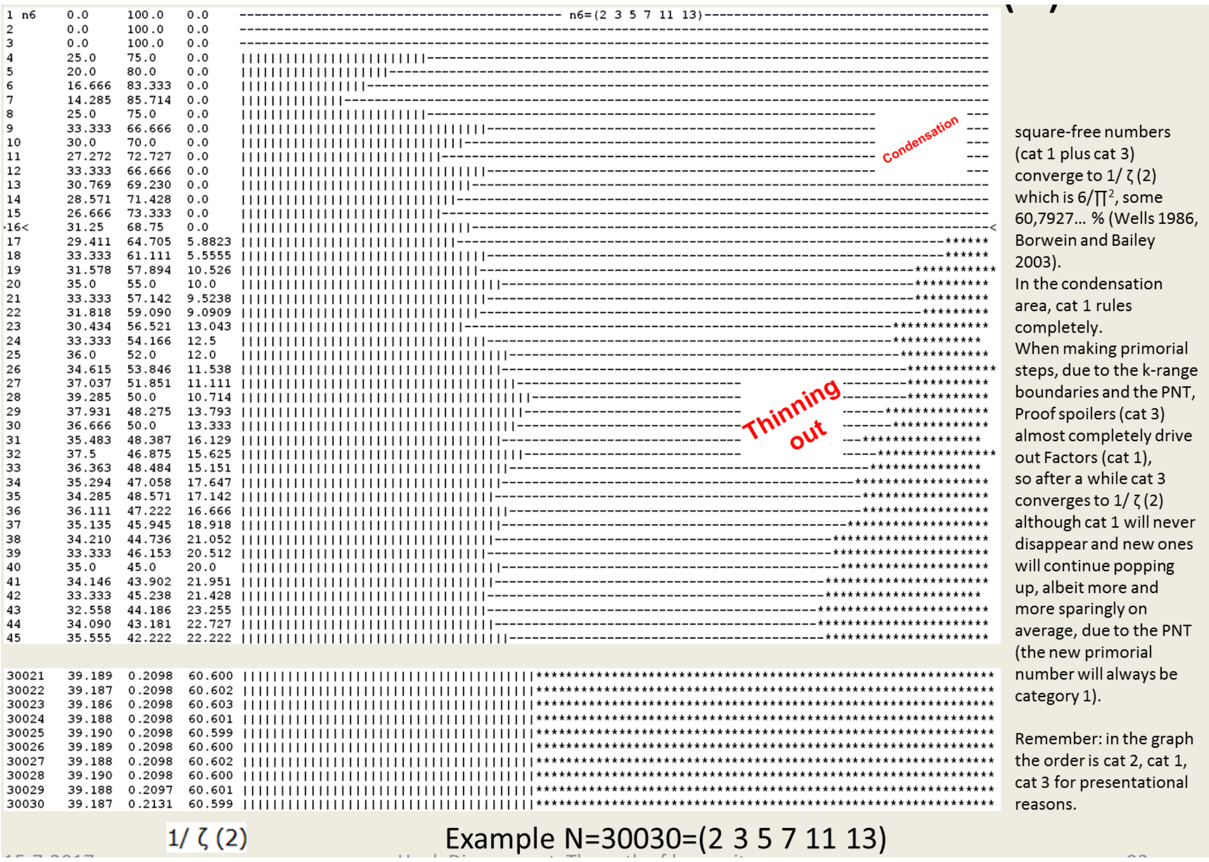
Note that the categories are presented in the order cat 2, cat 1 and cat 3 (from left to right), as this shows the disappearance of cat 3 numbers in a nice way.

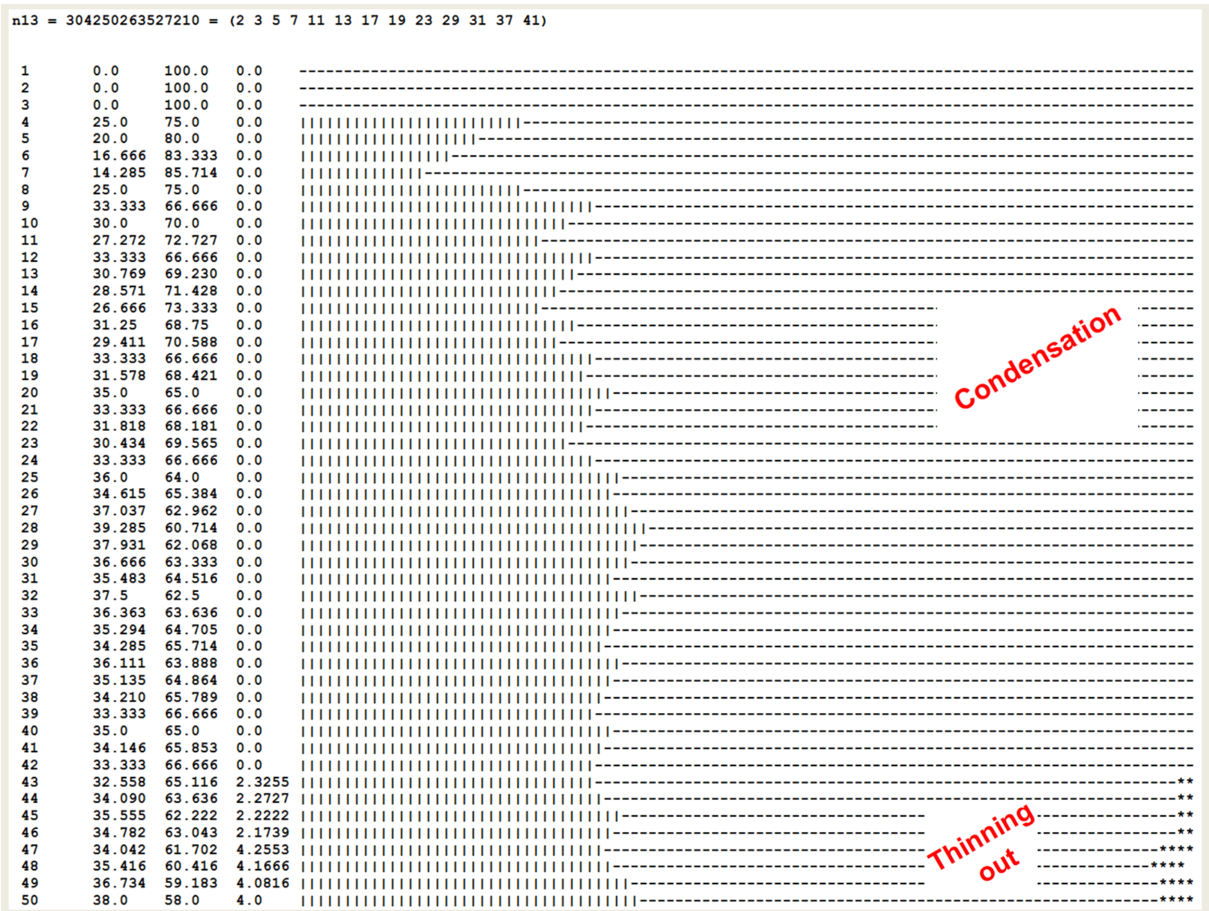


Example N=30=(2 3 5)



Example N=210=(2 3 5 7)


n6=(2 3 5 7 11 13)
Condensation
Thinning out
square-free numbers (cat 1 plus cat 3) converge to 1/ ζ (2) which is 6/∏², some 60,7927… % (Wells 1986, Borwein and Bailey 2003).
In the condensation area, cat 1 rules completely.
When making primorial steps, due to the k-range boundaries and the PNT, Proof spoilers (cat 3) almost completely drive out Factors (cat 1), so after a while cat 3 converges to 1/ ζ (2) although cat 1 will never disappear and new ones will continue popping up, albeit more and more sparingly on average, due to the PNT (the new primorial number will always be category 1).
Remember: in the graph the order is cat 2, cat 1, cat 3 for presentational reasons.
1/ ζ (2)
Example N=30030=(2 3 5 7 11 13)


n13 = 304250263527210 = (2 3 5 7 11 13 17 19 23 29 31 37 41)
Condensation
Thinning out

Epilogue

This paper represents the thoughts and considerations which make me believe that the Riemann hypothesis must be true. I recall: I am neither a mathematician nor a physicist. I do admire these disciplines, as I do admire any serious area of committed human endeavour, but I hardly use a mathematical or physical style of arguing or reasoning myself. I do not claim proficiency in either of these fields. Professionally, I am best characterized as a *reflective pragmatist* who uses a broad spectrum of scientific insights. In addition to this, I value and enjoy philosophizing and theorizing. However, what inspires me most and *always* remains my focal point is improving societal practice itself. This paper heavily rests on my forthcoming book: *The path of humanity*. The Riemann hypothesis, Recursive Perspectivism and The path of humanity share common grounds, and for this very reason I've hesitantly entered the domain of number theory and the Riemann hypothesis from the vantage point of Recursive Perspectivism. Hesitantly, as I am well aware of the significance of the Riemann hypothesis and my mathematical and physical proficiency level: I have very often felt like a fish out of the water.

It may be clear: I think the Riemann hypothesis holds water, and I have written down my ideas for a proof, albeit in a perhaps unconventional manner. I am not sure (and I sometimes even doubt) whether the exposé so far would impress, or even make sense to specialists like professional mathematicians or physicists at a first glance. Recursive Perspectivism is highly multi-disciplinary, and many disciplinary professionals are not. Recursive Perspectivism requires a willingness to articulate and reconsider philosophical premises of our understanding of our societies, and in my experience many disciplinary specialists are not willing to do so. In addition, at places my arguments may very well be error-ridden, and I am very likely to cut corners from a mathematical point of view. Reading and appreciating this paper will require an open and forgiving mind.

But *grosso modo* I think that my arguments for thinking why the Riemann hypothesis is true are written down clear enough for the time being. I hope they will help in giving attention to Recursive Perspectivism and its implications for human development and societal innovation, the main themes of the forthcoming book. All three the Riemann hypothesis, Recursive Perspectivism and The path of humanity are built on prime numbers, and this has important implications for human development and societal innovation. It may be of help in finding better ways to a more sustainable, a better future. For these very reasons I've written this paper, the presentation and the book. Of course I am open to comments and corrections. But most of all, I would like to suggest that you read the presentation and "The path of humanity" (Dutch version in the beginning of 2018, English version May or June 2018), and engage in the explorations they present. Thank you for your attention.

Henk Diepenmaat

Zeist, The Netherlands, 31-12-2017
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About Henk Diepenmaat

Henk Diepenmaat, PhD MSc MSc (1962), is the director of Actors Process Management Ltd, a small consulting firm. He lives on the edge of a forest in Zeist, a village in the centre of the Netherlands, with his wife and two young adult sons. His primary expertise is in *multi-actor process management* and *societal innovation*. He is a professorial fellow at the International Centre for Integrated Assessment and Sustainable Development (ICIS) at Maastricht University in the Netherlands.

Henk has thirty years' experience in getting a deep understanding of complex playing fields and working purposefully in them. Through his work, he creates value for businesses, governments, citizens, consumers, and other actors. Professionally, Henk is best characterized as a *reflective practitioner* who uses a broad spectrum of scientific insights. In addition to this, he values and enjoys philosophizing and theorizing. However, what inspires him most and *always* remains his focal point is improving societal practice itself. In order to contribute to societal improvement, Henk advises and supports businesses, governments, and other organizations in strategy, innovation, and business or policy development and deployment. He is available for in-company and open workshops, lectures, and training in multi-actor process management and innovation. In these activities, he makes a point of using (and stretching) the actual interests and situations of the participants' work and lives as ingredients for case studies so as to add relevance and meaning in order to facilitate maximum learning.

Henk has written several books, including a handbook *Multi-Actor Process Management in Theory and Practice* (in Dutch, English in preparation), and his magnum opus *The Path of Humanity: Societal Innovation for the World of Tomorrow* (in press).

Henk holds two master's degrees (chemistry, Radboud University, Nijmegen, the Netherlands, and environmental sciences, Wageningen University, the Netherlands; knowledge and information technology, Middlesex University, London). His PhD research was about the theory and practice of multi-actor processes (Faculties of Spatial Sciences and Social Sciences, University of Amsterdam).

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References

In this paper I predominantly use well established and ancient mathematics, which can readily be found on the internet. I am in debt of forums like Mathworld.wolfram.com, Physicsforums.com, Numberphile and Wikipedia, for increasing my general understanding of many of the matters at hand. I also learned a lot from John Derbyshire's "Prime Obsession" (2004 Plume, Penguin Group). I have experienced on many occasions that media like these are not taken very seriously by professional disciplinary scientists. Many read them secretly, but avoid mentioning this. Although I am a fan of the traditional scientific mind set, I pity this. Disciplinary and scientific specialists contribute to society in important ways, and will do so in our future. It is, however, my sincere belief that we humans need to cross boundaries between sciences, disciplines, cultures and experiential worlds to a far, far larger degree than we do at this very moment. I try to live and work in line with this belief. The book "The path of humanity" is *both* a result of *and* a plea for this attitude. As a consequence, I am an intense user of media that popularize serious science of the past and the present in a professional way, and I commend these media for the good and important bridging work they are doing. Chapeau!

I explicitly refer to one serious scientific mathematical source:

Edwards, H.M. (1974), **Riemann's Zeta Function**, Dover Publications, Inc., Mineola, New York.

In addition I refer to the following scientific sources (they provide some background):

Odlyzko, A. M.; te Riele, H. J. J. (1985), **Disproof of the Mertens conjecture**, Journal für die reine und angewandte Mathematik, **357**: 138–160.

Bonse, H. (1907), "Über eine bekannte Eigenschaft der Zahl 30 und ihre Verallgemeinerung", Archiv der Mathematik und Physik 3 (12): 292–295.

In addition to this, the paper rests heavily on the following sources:

Diepenmaat, H.B. (1997), Trinity, **Model-based support for Multi-actor Problem Solving**, applied to environmental problems, PhD Thesis, University of Amsterdam.

The book (part 5 of the series Society in perspective):
Diepenmaat, Henk (2018), **The path of humanity**: societal innovation for the world of tomorrow, Parthenon Publishers, Almere, The Netherlands, due may-june 2018.

The presentation (Actors Publication 5C):
Diepenmaat, Henk, Relations between societal innovation and number theory: **Dicey proofs of the Riemann hypothesis**, presentation Actors Process Management, Zeist, The Netherlands, December 31, 2017.